Q-Learning
and Pontryagin's Minimum Principle

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Research support: NSF: ECS-0523620
AFOSR: FA9550-09-1-0190

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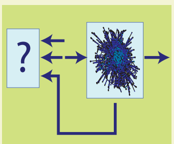
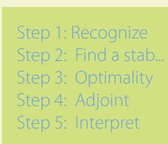
Department of Electrical and Computer Engineering

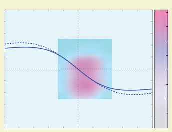
Outline



Coarse models - what to do with them?



Q-learning for nonlinear state space models



Example: Local approximation

Example: Decentralized control

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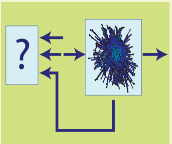

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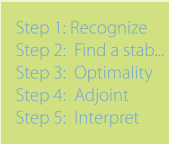

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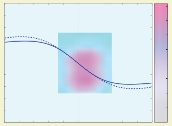
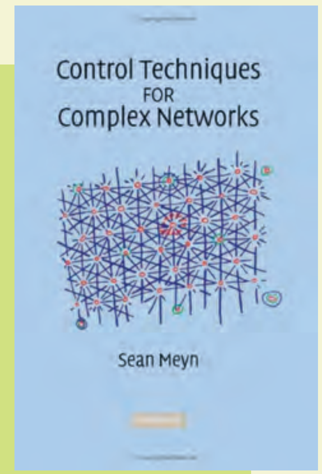


Example: Local approximation

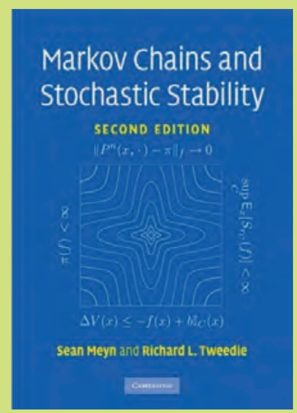
Example: Decentralized control

Coarse Models: *A rich collection of model reduction techniques*

Many of today's participants have contributed to this research.
A biased list:

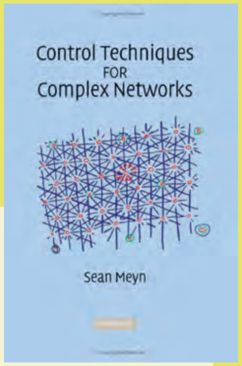


- *Fluid models*: Law of Large Numbers scaling, most likely paths in large deviations
- *Workload relaxation* for networks
  Heavy-traffic limits



- *Clustering*: spectral graph theory
  Markov spectral theory
- *Singular perturbations*
- *Large population limits*: Interacting particle systems

Workload Relaxations



An example from CTCN:

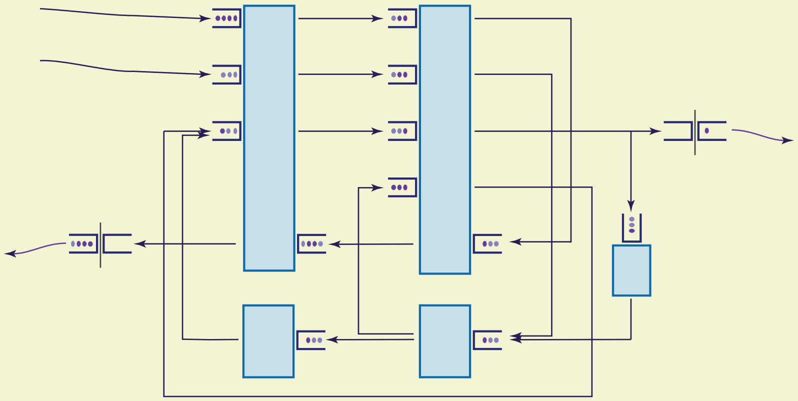


Figure 7.1: Demand-driven model with routing, scheduling, and re-work.

Workload at two stations evolves as a two-dimensional system
Cost is projected onto these coordinates:

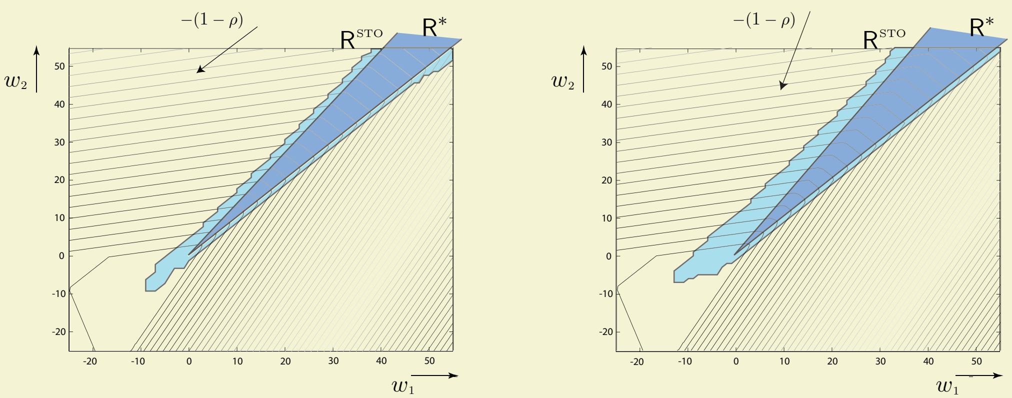
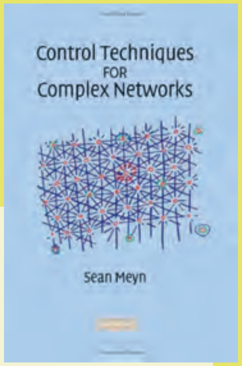


Figure 7.2: Optimal policies for two instances of the network shown in Figure 7.1. In each figure the optimal stochastic control region $\mathsf{R}^{\mathrm{STO}}$ is compared with the optimal region R^* obtained for the two dimensional fluid model.

Optimal policy for relaxation = hedging policy for full network

Workload Relaxations and Simulation



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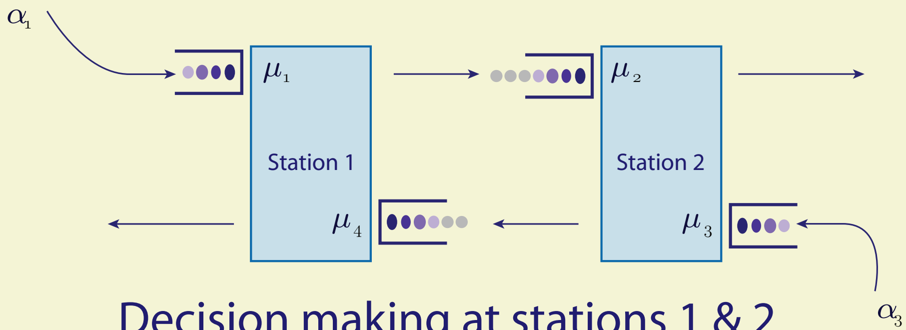
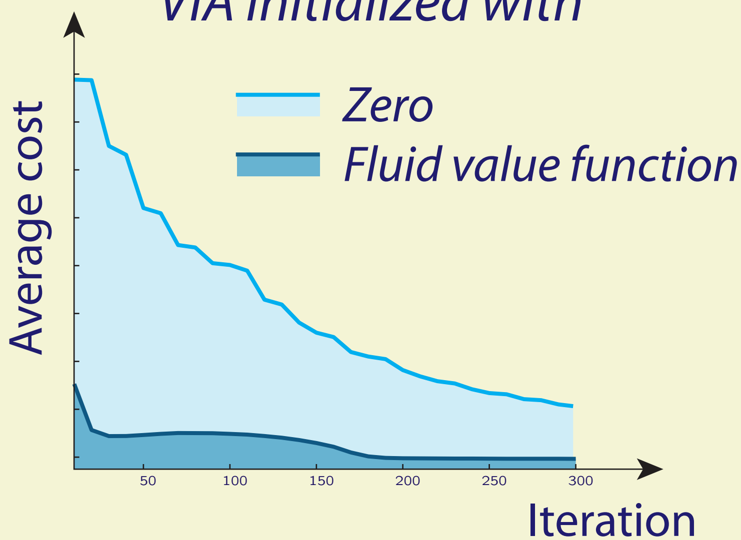


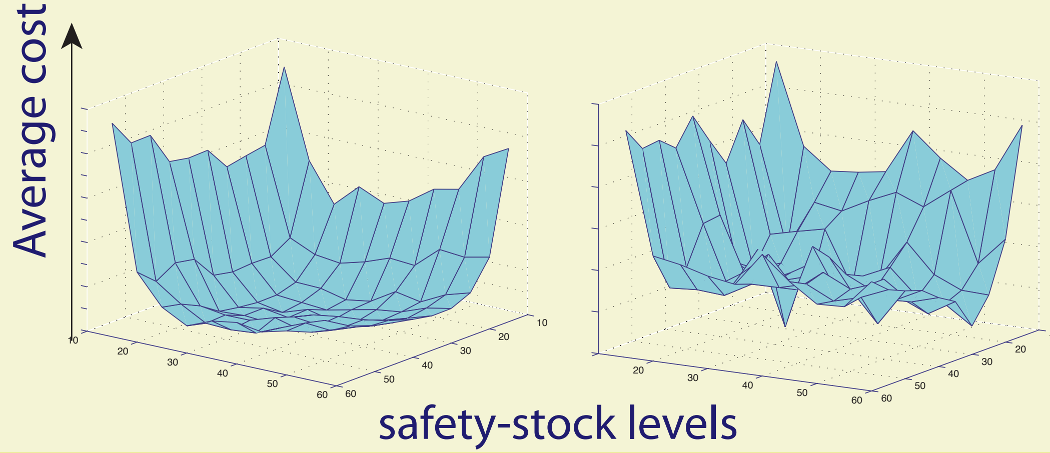
Decision making at stations 1 & 2
e.g., setting safety-stock levels

DP and simulations accelerated
using *fluid value function* for *workload relaxation*

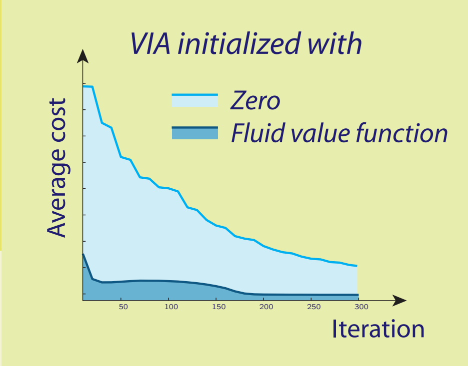
VIA initialized with



Simulated mean with
and without control variate:



What To Do With a Coarse Model?



Setting: we have qualitative or partial quantitative insight regarding optimal control

The network examples relied on specific network structure

What about other models?

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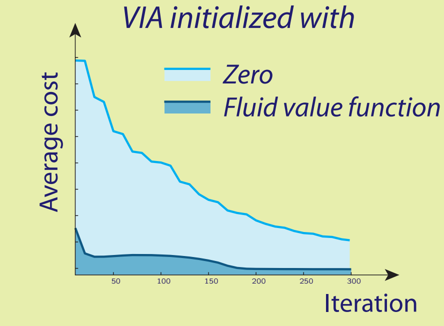

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What about other models?

An answer lies in a new formulation of Q-learning

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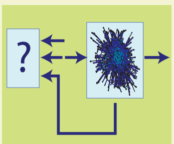

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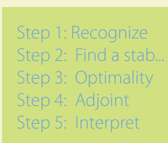
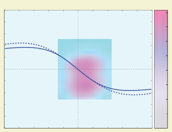


Q-learning for nonlinear state space models

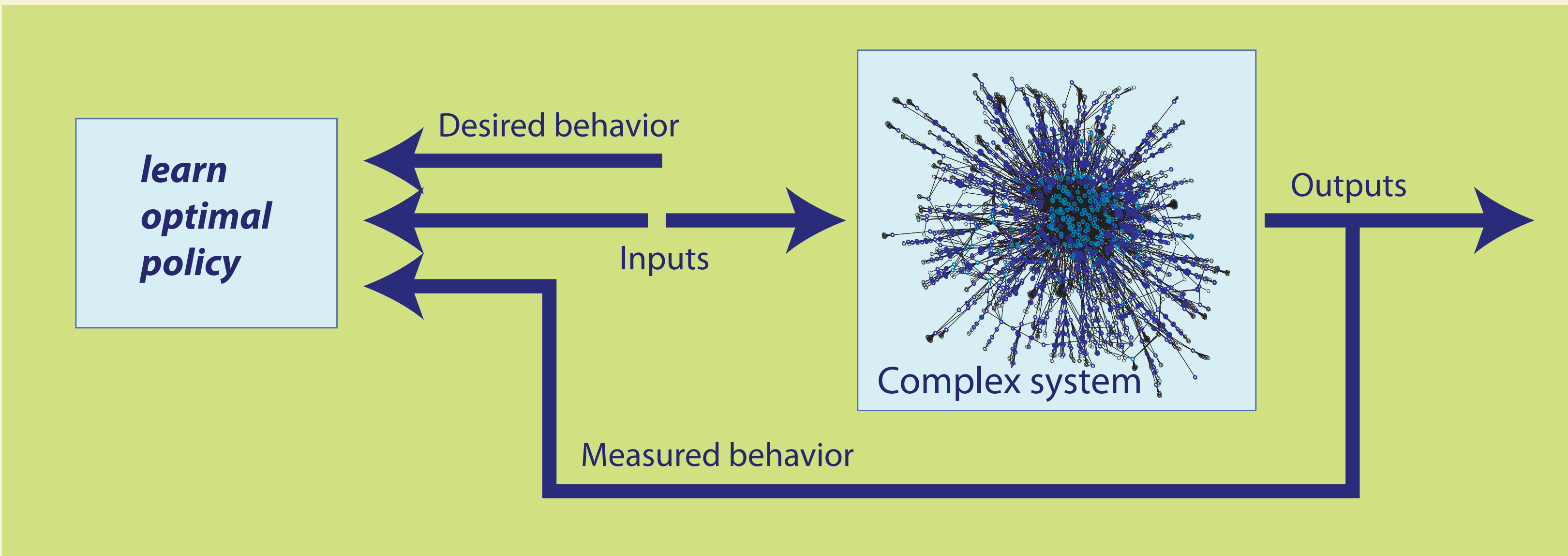


Example: Local approximation

Example: Decentralized control

What is Q learning?

Identify optimal policy based on observations:



Watkin’s 1992 formulation applied to finite state space MDPs

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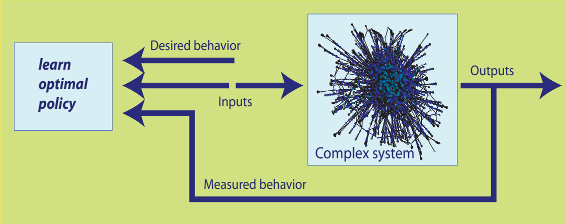

Watkin's 1992 formulation applied to finite state space MDPs

Watkins and P. Dayan, 1992

Goal: Find the best approximation to dynamic programming equations over a parameterized class, based on observations using a non-optimal policy.

Watkin's algorithm known to be effective only for

- Finite state-action space
- Complete parametric family

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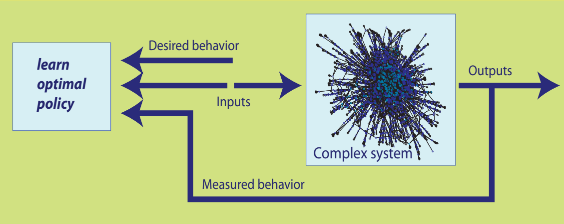

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Finite state-action space
Complete parametric family

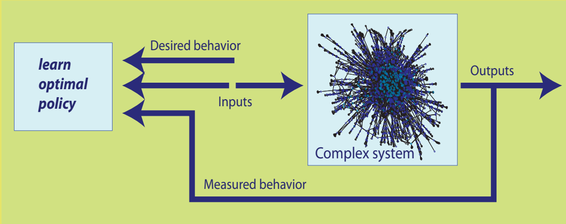
Extensions: when cost depends on control, but dynamics are oblivious

Duff, 1995
Tsitsiklis and Van Roy, 1999
Yu and Bertsekas, 2007

Approach: Similar to *differential dynamic programming*

Differential dynamic programming
D. H. Jacobson and D. Q. Mayne
American Elsevier Pub. Co. 1970

What is Q learning?



Watkin's 1992 formulation applied to finite state space MDPs

This lecture:

Deterministic formulation: Nonlinear system on Euclidean space,

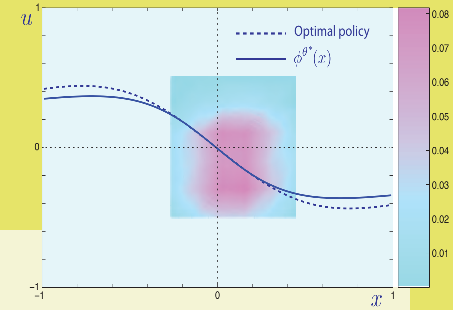
$$\frac{d}{dt}x(t) = f(x(t), u(t)), \qquad t \geq 0$$

Infinite-horizon discounted cost criterion,

$$J^*(x) = \inf \int_0^\infty e^{-\gamma s} c(x(s), u(s))\, ds, \qquad x(0) = x$$

with c a non-negative cost function.

What is Q learning?



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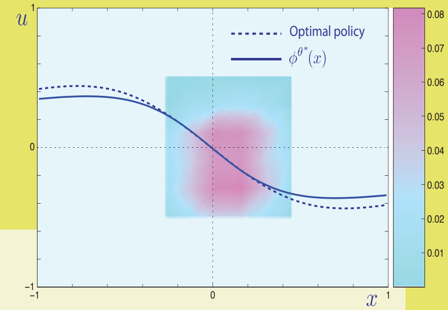
Differential generator: For any smooth function h,

$$\mathcal{D}_u h\,(x) := (\nabla h\,(x))^{T} f\,(x, u)$$

What is Q learning?



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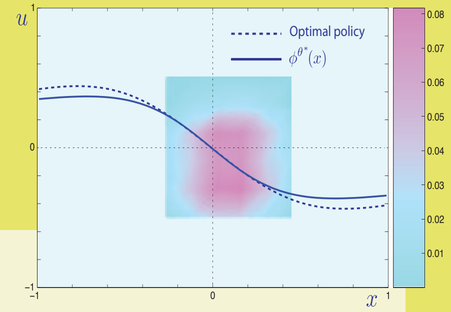

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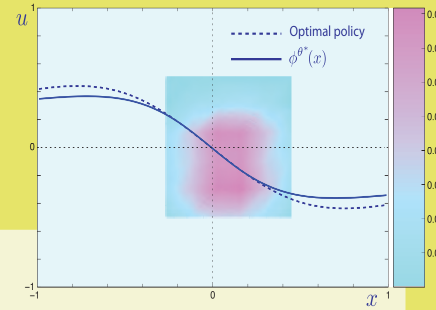
The *Q-function* of Q-learning is this function of two variables

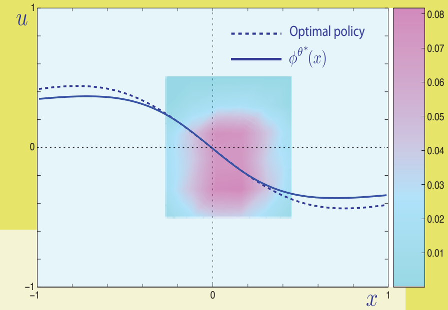
Q learning - Steps towards an algorithm



Sequence of five steps:

Step 1: Recognize fixed point equation for the Q-function
Step 2: Find a stabilizing policy that is ergodic
Step 3: Optimality criterion - minimize Bellman error
Step 4: Adjoint operation
Step 5: Interpret and simulate!

Q learning - Steps towards an algorithm



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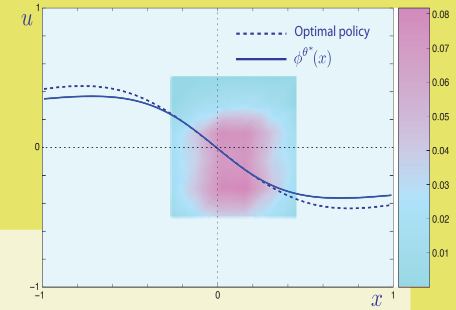
Goal - seek the best approximation,
within a parameterized class

$$H^\theta(x,u) = \theta^T \psi(x,u), \qquad \theta \in \mathbb{R}^d$$

Q learning - Steps towards an algorithm



Step 1: Recognize fixed point equation for the Q-function

Q-function: $$H^*(x,u) = c(x,u) + \mathcal{D}_u J^*(x)$$

Its minimum: $$\underline{H}^*(x) := \min_{u \in \mathsf{U}} H^*(x,u) = \gamma J^*(x)$$

Fixed point equation:

$$\mathcal{D}_u \underline{H}^*(x) = -\gamma(c(x,u) - H^*(x,u))$$

Step 1: Recognize fixed point equation for the Q-function
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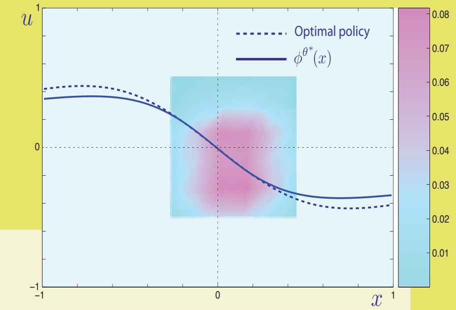
Q learning - Steps towards an algorithm



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Fixed point equation:

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Key observation for learning: For any input-output pair,

$$\mathcal{D}_u \underline{H}^*(x) = \frac{d}{dt}\underline{H}^*(x(t))\Big|_{\substack{x=x(t)\\ u=u(t)}}$$

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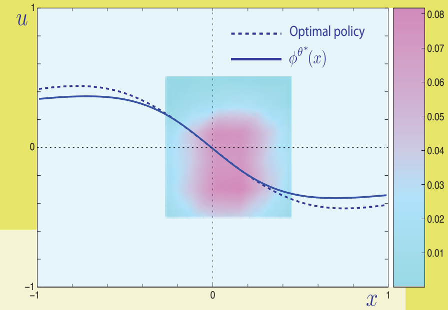

Q learning - LQR example

Linear model and quadratic cost,

Cost: $$c(x,u) \;=\; \tfrac{1}{2}x^TQx \;+\; \tfrac{1}{2}u^TRu$$

Q-function: $$H^*(x) = c(x,u) + (Ax + Bu)^T P^* x$$

Solves Riccatti eqn

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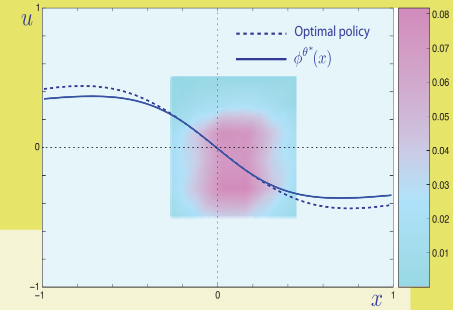

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Q-function: $$H^*(x) = c(x,u) + (Ax+Bu)^{\scriptscriptstyle T}P^*x$$

Solves Riccati eqn

Q-function approx:

$$H^\theta(x,u) = c(x,u) + \tfrac{1}{2}\sum_{i=1}^{d_x}\theta_i^x x^{\scriptscriptstyle T}E^i x + \sum_{j=1}^{d_{xu}}\theta_j^x x^{\scriptscriptstyle T}F^i u$$

Minimum:

$$\underline{H}^\theta(x) = \tfrac{1}{2}x^{\scriptscriptstyle T}\Big(Q + E^\theta - F^{\theta^{\scriptscriptstyle T}}R^{-1}F^\theta\Big)\,x$$

Minimizer:

$$u^\theta(x) = \phi^\theta(x) = -R^{-1}F^\theta x$$

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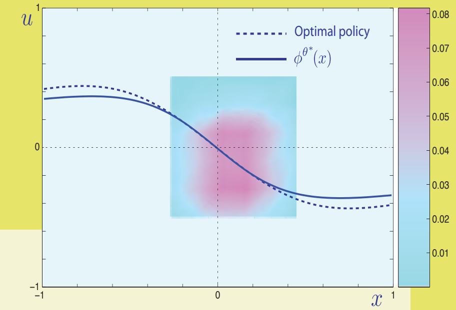
Q learning - Steps towards an algorithm



Step 2: Stationary policy that is ergodic?

Assume the LLN holds for continuous functions

$$F\colon \mathbb{R}^{\ell} \times \mathbb{R}^{\ell_u} \to \mathbb{R}$$

As $T \to \infty$,

$$\frac{1}{T}\int_0^T F(x(t),u(t))\,dt \longrightarrow \int_{\mathsf{X}\times\mathsf{U}} F(x,u)\,\varpi(dx,du)$$

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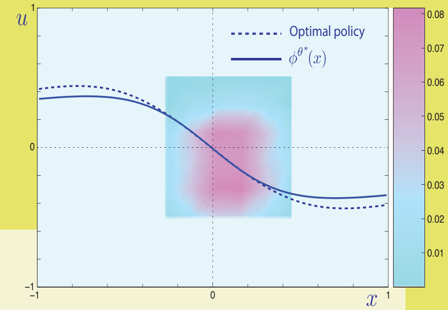

Step 2: Stationary policy that is ergodic?

Suppose for example the input is scalar, and the system is *stable*

[Bounded-input/Bounded-state]

Can try a linear combination of sinusouids

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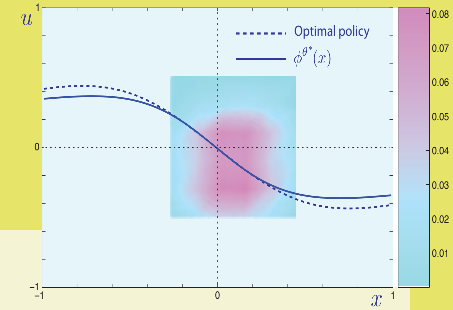

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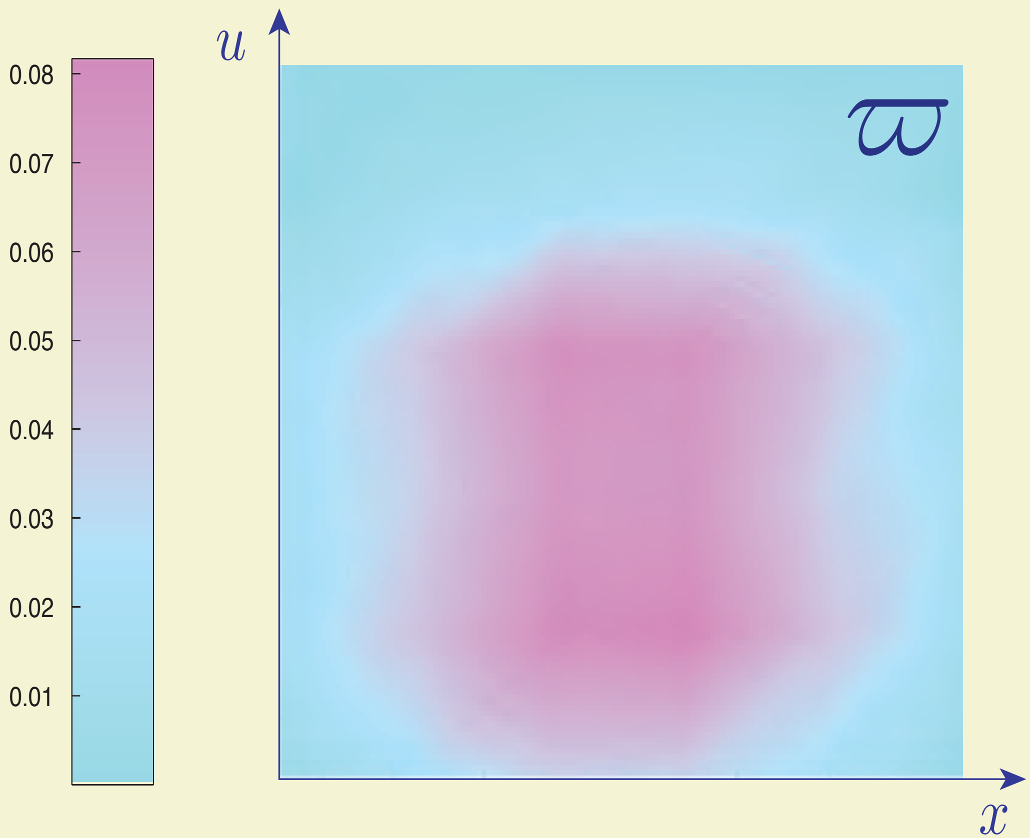

Can try a linear combination of sinusoids

$$u(t) = A(\sin(t) + \sin(\pi t) + \sin(et))$$

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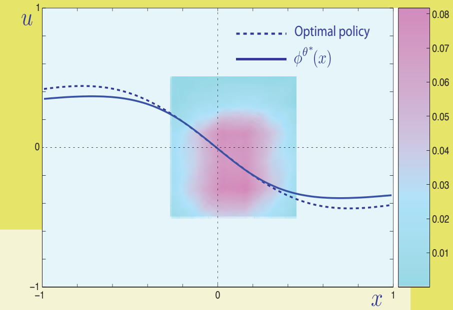

Step 3: Bellman error

$$\mathcal{L}^{\theta}(x,u) \quad := \quad \mathcal{D}_u \underline{H}^{\theta}(x) + \gamma(c - H^{\theta}), \qquad \theta \in \mathbb{R}^d$$

Based on observations, minimize the mean-square Bellman error:

$$\mathcal{E}_{\mathrm{Bell}}(\theta) \quad := \quad \int [\mathcal{L}^{\theta}]^2 \varpi(dx,du) \;:=\; \langle \mathcal{L}^{\theta},\, \mathcal{L}^{\theta} \rangle_{\varpi}$$

First order condition for optimality: $\langle \mathcal{L}^{\theta}, \mathcal{D}_u \underline{\psi}_i^{\theta} - \gamma \psi_i^{\theta} \rangle_{\varpi} = 0$

with $\underline{\psi}_i^{\theta}(x) = \psi_i^{\theta}(x, \phi^{\theta}(x))$,

$1 \le i \le d$

$$\mathcal{D}_u \underline{H}^{\theta}(x) = \frac{d}{dt} \underline{H}^{\theta}(x(t)) \Big|_{\substack{x=x(t)\\ u=u(t)}}$$

$$\mathcal{D}_u \underline{\psi}_i^{\theta}(x) = \frac{d}{dt} \underline{\psi}_i^{\theta}(x(t)) \Big|_{\substack{x=x(t)\\ u=u(t)}}$$

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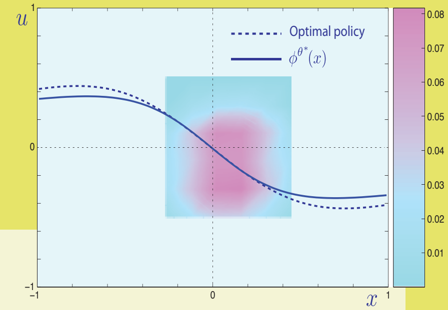
Q learning - Convex Reformulation



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$$\mathcal{L}^\theta(x,u) \quad := \quad \mathcal{D}_u G^\theta(x) + \gamma(c - H^\theta), \qquad \theta \in \mathbb{R}^d$$

$$G^\theta(x) \le H^\theta(x,u), \qquad \text{all } x,\ u$$

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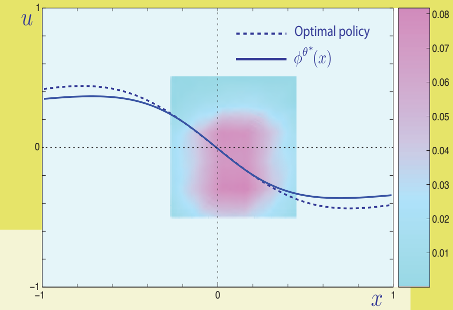

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Solves Riccati eqn

Q-function approx:

$$H^\theta(x,u) = c(x,u) + \frac{1}{2}\sum_{i=1}^{d_x} \theta_i^x x^T E^i x + \sum_{j=1}^{d_{xu}} \theta_j^x x^T F^i u$$

Approximation to minimum

$$G^\theta(x) = \tfrac{1}{2}x^T G^\theta x$$

Minimizer:

$$u^\theta(x) = \phi^\theta(x) = -R^{-1}F^\theta x$$

Step 1: Recognize fixed point equation for the Q-function
Step 2: Find a stabilizing policy that is ergodic
Step 3: Optimality criterion - minimize Bellman error
Step 4: Adjoint operation
Step 5: Interpret and simulate!

Q learning - Steps towards an algorithm

$$\cancel{\begin{aligned}\mathcal{D}_u \underline{H}^\theta(x) &= \tfrac{d}{dt}\underline{H}^\theta(x(t)) \\ \mathcal{D}_u \underline{\psi}_i^\theta(x) &= \tfrac{d}{dt}\underline{\psi}_i^\theta(x(t))\end{aligned}}$$

Step 4: Causal smoothing to avoid differentiation

For any function of two variables, $g : \mathbb{R}^\ell \times \mathbb{R}^{\ell_w} \to \mathbb{R}$
Resolvent gives a new function,

$$R_\beta g\,(x, w) = \int_0^\infty e^{-\beta t} g(x(t), \xi(t))\, dt$$

Skip to examples

$$\cancel{\mathcal{D}_u \underline{H}^\theta(x) = \tfrac{d}{dt}\underline{H}^\theta(x(t))}$$

$$\cancel{\mathcal{D}_u \underline{\psi}_i^\theta(x) = \tfrac{d}{dt}\underline{\psi}_i^\theta(x(t))}$$

Q learning - Steps towards an algorithm

Step 4: Causal smoothing to avoid differentiation

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Resolvent gives a new function,

$$R_\beta g\,(x, w) = \int_0^\infty e^{-\beta t} g(x(t), \xi(t))\, dt \;, \qquad \beta > 0$$

controlled using the nominal policy

$$u(t) = \phi(x(t), \xi(t)), \qquad t \geq 0$$

stabilizing & ergodic

Q learning - Steps towards an algorithm

$$\cancel{\mathcal{D}_u \underline{H}^\theta(x) = \frac{d}{dt}\underline{H}^\theta(x(t))}$$

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Resolvent equation:

$$R_\beta \mathcal{D} = \beta R_\beta - I$$

Q learning - Steps towards an algorithm

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Resolvent equation:

$$R_\beta \mathcal{D} = \beta R_\beta - I$$

Smoothed Bellman error:

$$\begin{aligned}\mathcal{L}^{\theta,\beta} &= R_\beta \mathcal{L}^\theta \\ &= [\beta R_\beta - I]\underline{H}^\theta + \gamma R_\beta (c - H^\theta)\end{aligned}$$

Q learning - Steps towards an algorithm

$$\mathcal{D}_u \underline{H}^\theta\,(x) = \tfrac{d}{dt}\underline{H}^\theta(x(t))$$

$$\mathcal{D}_u \underline{\psi}_i^\theta\,(x) = \tfrac{d}{dt}\underline{\psi}_i^\theta(x(t))$$

Smoothed Bellman error:

$$\mathcal{E}_\beta(\theta) \;:=\; \tfrac{1}{2}\|\mathcal{L}^{\theta,\beta}\|^2_\varpi$$

$$\nabla\mathcal{E}_\beta(\theta) = \langle \mathcal{L}^{\theta,\beta}, \nabla_\theta \mathcal{L}^{\theta,\beta}\rangle_\varpi$$

$=$ *zero* at an optimum

Step 4: Causal smoothing to avoid differentiation

Q learning - Steps towards an algorithm

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$=$ *zero* at an optimum

Involves terms of the form $\langle R_\beta g, R_\beta h\rangle$

Step 4: Causal smoothing to avoid differentiation

Q learning - Steps towards an algorithm

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Adjoint operation:

$$R_\beta^\dagger R_\beta = \frac{1}{2\beta}\left(R_\beta^\dagger + R_\beta\right)$$

$$\langle R_\beta g, R_\beta h\rangle = \frac{1}{2\beta}\big(\langle g, R_\beta^\dagger h\rangle + \langle h, R_\beta^\dagger g\rangle\big)$$

Step 4: Causal smoothing to avoid differentiation

$$\mathcal{D}_u \underline{H}^\theta(x) = \frac{d}{dt}\underline{H}^\theta(x(t))$$

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Q learning - Steps towards an algorithm

Smoothed Bellman error: $\mathcal{E}_\beta(\theta) \;:=\; \frac{1}{2}\|\mathcal{L}^{\theta,\beta}\|^2_\varpi$

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Adjoint operation:

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$$\langle R_\beta g, R_\beta h\rangle = \frac{1}{2\beta}\big(\langle g, R_\beta^\dagger h\rangle + \langle h, R_\beta^\dagger g\rangle\big)$$

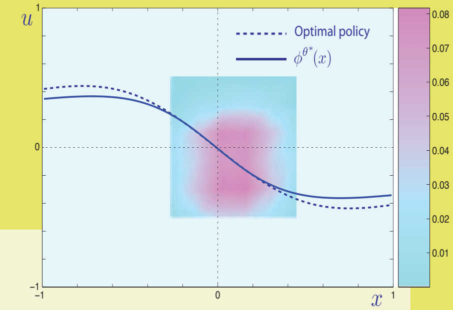
Adjoint realization: *time-reversal*

$$R_\beta^\dagger g\,(x,w) = \int_0^\infty e^{-\beta t}\mathsf{E}_{x,\,w}[g(x^\circ(-t),\xi^\circ(-t))]\,dt$$

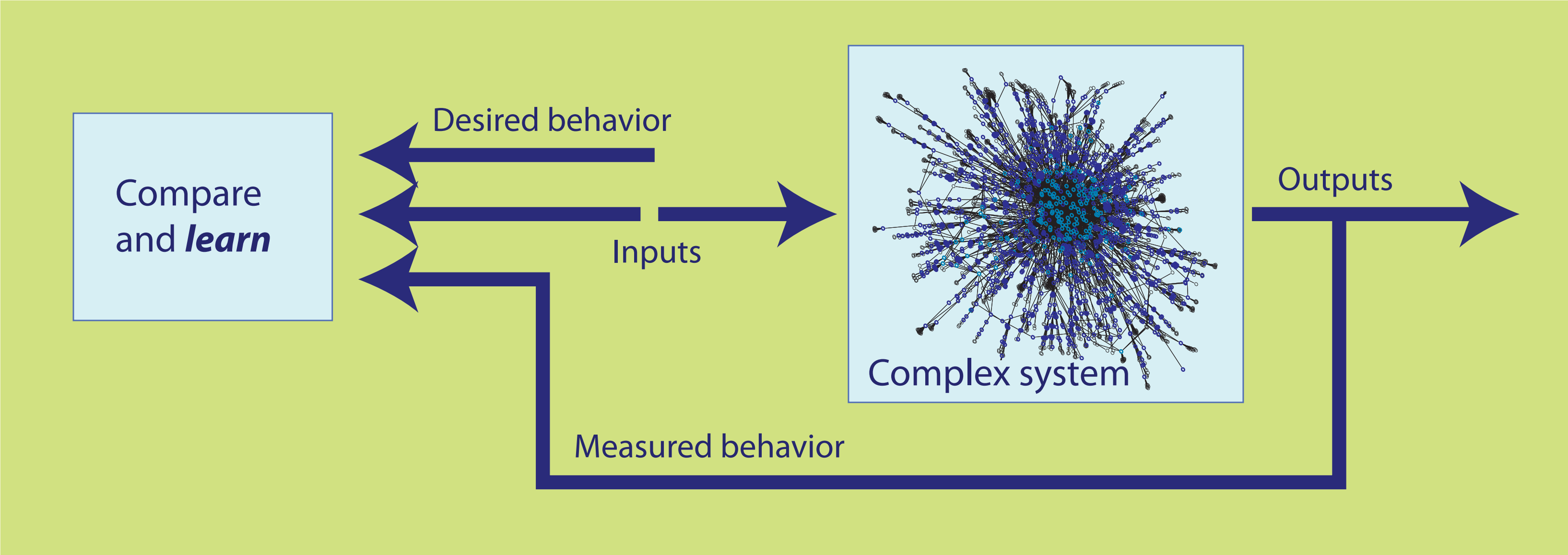
expectation conditional on $x^\circ(0) = x,\ \xi^\circ(0) = w.$

Step 4: Causal smoothing to avoid differentiation

Q learning - Steps towards an algorithm



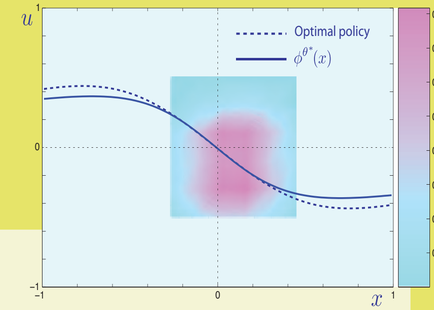
After Step 5: Not quite adaptive control:



Ergodic input applied

Step 1: Recognize fixed point equation for the Q-function
Step 2: Find a stabilizing policy that is ergodic
Step 3: Optimality criterion - minimize Bellman error
Step 4: Adjoint operation
Step 5: Interpret and simulate!

Q learning - Steps towards an algorithm



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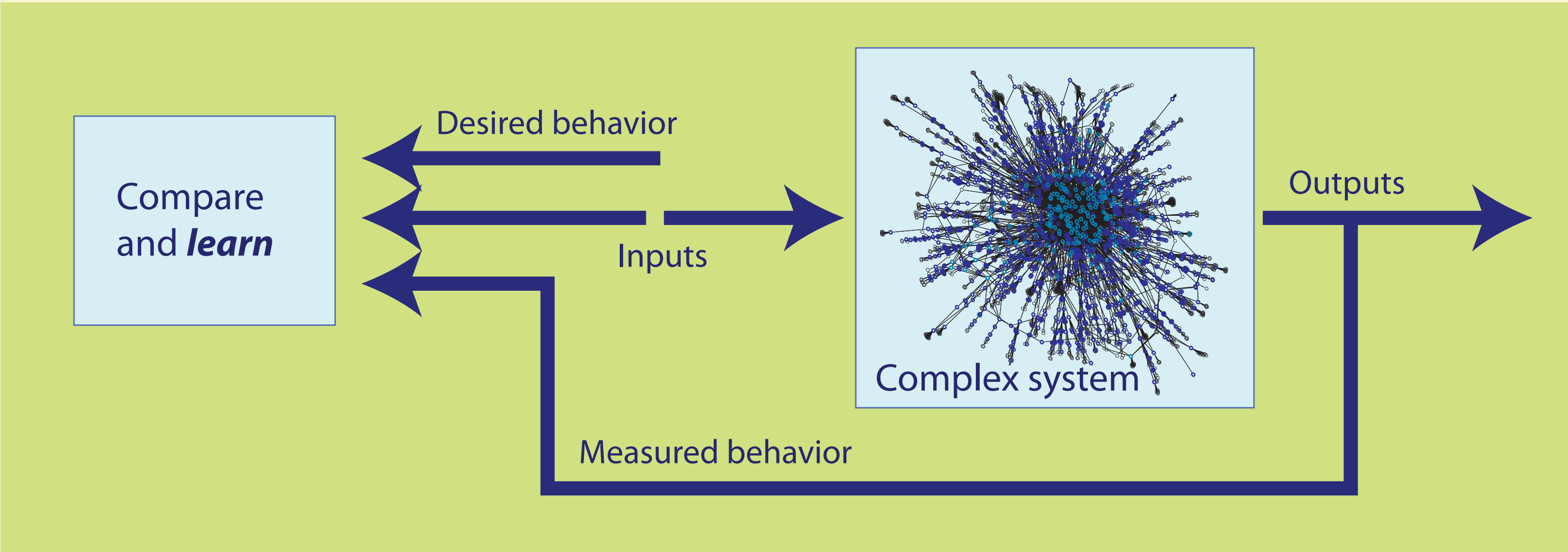


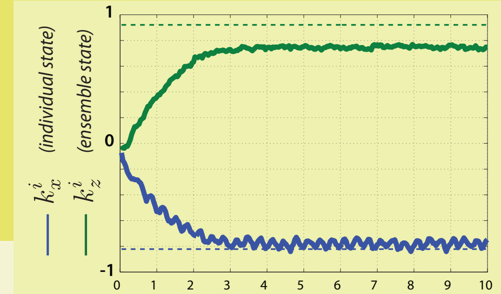
Ergodic input applied

Based on observations minimize the mean-square Bellman error:

$$\mathcal{E}_{\rm Bell}(\theta) \quad := \quad \int \big[\mathcal{L}^\theta\big]^2 \varpi(dx, du)$$

$$\mathcal{L}^\theta(x,u) \quad := \quad \mathcal{D}_u \underline{H}^\theta(x) + \gamma(c - H^\theta)\,, \qquad \theta \in \mathbb{R}^d$$

Deterministic Stochastic Approximation



Gradient descent:

$$\frac{d}{dt}\theta = -\varepsilon \langle \mathcal{L}^\theta, \mathcal{D}_u \nabla_\theta \underline{H}^\theta - \gamma \nabla_\theta H^\theta \rangle_\varpi$$

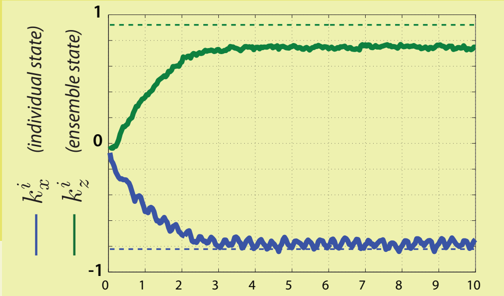
Converges* to the minimizer of the mean-square Bellman error:

$$\begin{aligned} \mathcal{E}_{\mathrm{Bell}}(\theta) \quad &:= \quad \int \big[\mathcal{L}^\theta\big]^2 \varpi(dx, du) \\ \mathcal{L}^\theta(x,u) \quad &:= \quad \mathcal{D}_u \underline{H}^\theta(x) + \gamma(c - H^\theta) \end{aligned}$$

$$\frac{d}{dt} h(x(t)) \Big|_{\substack{x = x(t) \\ w = \xi(t)}} = \mathcal{D}_u h(x)$$

* Convergence observed in experiments!
For a convex re-formulation of
the problem, see Mehta & Meyn 2009

Deterministic Stochastic Approximation



Stochastic Approximation

$$\frac{d}{dt}\theta = -\varepsilon_t \mathcal{L}_t^\theta \Big(\frac{d}{dt}\nabla_\theta \underline{H}^\theta\,(x^\circ(t)) - \gamma \nabla_\theta H^\theta(x^\circ(t), u^\circ(t))\Big)$$

$$\mathcal{L}_t^\theta := \frac{d}{dt}\underline{H}^\theta\,(x^\circ(t)) + \gamma(c(x^\circ(t), u^\circ(t)) - H^\theta(x^\circ(t), u^\circ(t)))$$

Gradient descent:

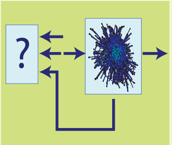
$$\frac{d}{dt}\theta = -\varepsilon \langle \mathcal{L}^\theta, \mathcal{D}_u \nabla_\theta \underline{H}^\theta - \gamma \nabla_\theta H^\theta \rangle_\varpi$$

Mean-square Bellman error:

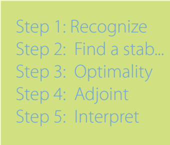
$$\begin{aligned} \mathcal{E}_{\text{Bell}}(\theta) &:= \int [\mathcal{L}^\theta]^2\, \varpi(dx, du) \\ \mathcal{L}^\theta(x,u) &:= \mathcal{D}_u \underline{H}^\theta\,(x) + \gamma(c - H^\theta) \end{aligned}$$

$$\frac{d}{dt} h(x(t)) \Big|_{\substack{x=x(t)\\ w=\xi(t)}} = \mathcal{D}_u h\,(x)$$

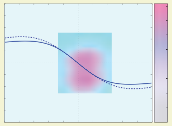
Outline



Coarse models - what to do with them?



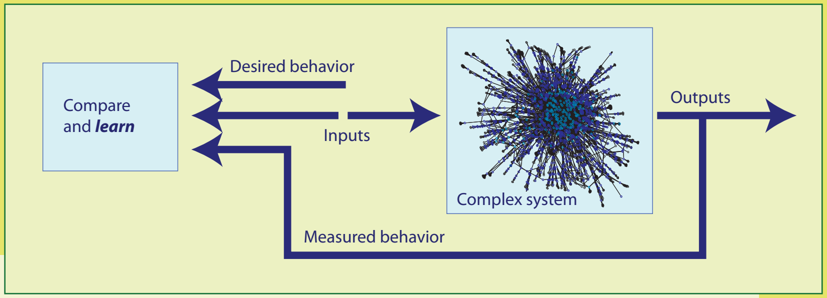
Q-learning for nonlinear state space models



Example: Local approximation

Example: Decentralized control

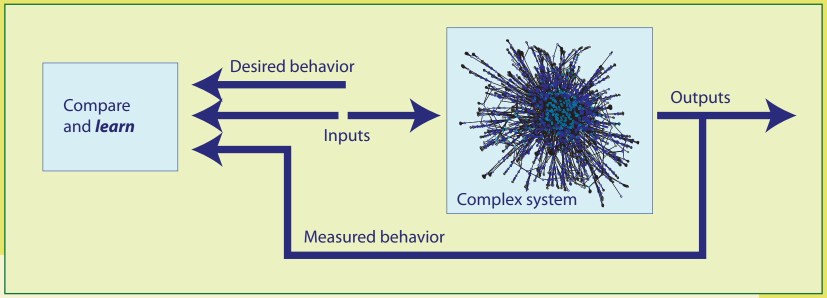
Q learning - Local Learning



Cubic nonlinearity:

$$\frac{d}{dt}x = -x^3 + u, \qquad c(x,u) = \frac{1}{2}x^2 + \frac{1}{2}u^2$$

Q learning - Local Learning

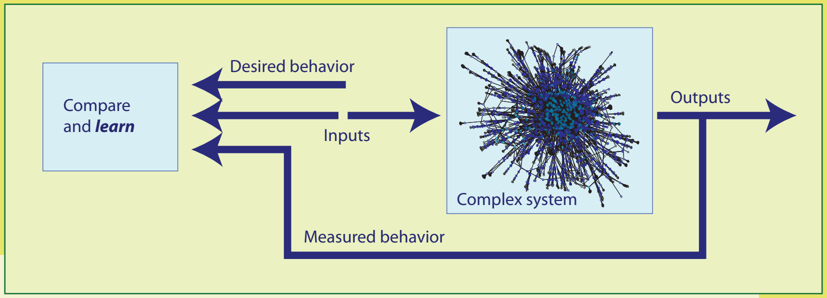


Cubic nonlinearity: $\frac{d}{dt}x = -x^3 + u, \qquad c(x,u) = \frac{1}{2}x^2 + \frac{1}{2}u^2$

HJB:

$$\min_u \Big(\tfrac{1}{2}x^2 + \tfrac{1}{2}u^2 + (-x^3 + u)\nabla J^*(x) \Big) = \gamma J^*(x)$$

Q learning - Local Learning



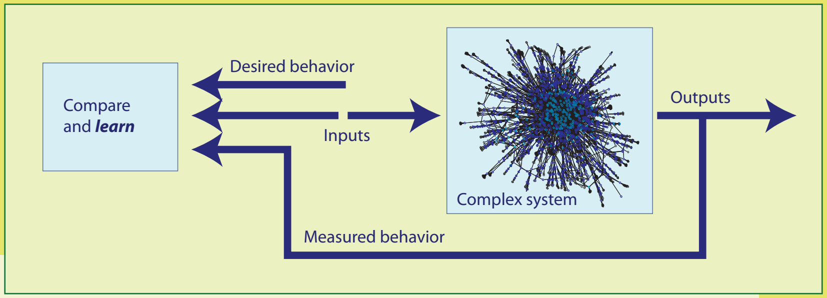
Cubic nonlinearity: $\frac{d}{dt}x = -x^3 + u, \qquad c(x,u) = \frac{1}{2}x^2 + \frac{1}{2}u^2$

HJB: $\min_u \Big(\frac{1}{2}x^2 + \frac{1}{2}u^2 + (-x^3 + u)\nabla J^*(x) \Big) = \gamma J^*(x)$

Basis:

$$H^\theta(x,u) = c(x,u) + \theta^x x^2 + \theta^{xu} \frac{x}{1+2x^2} u$$

Q learning - Local Learning



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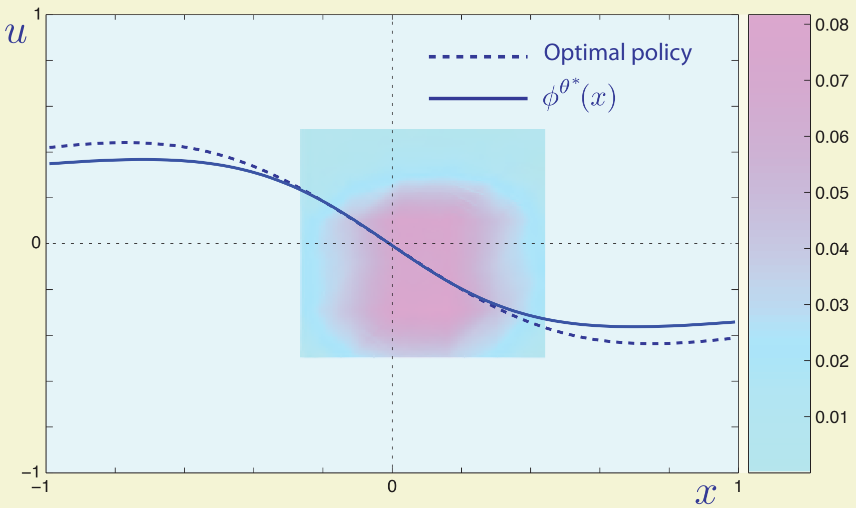
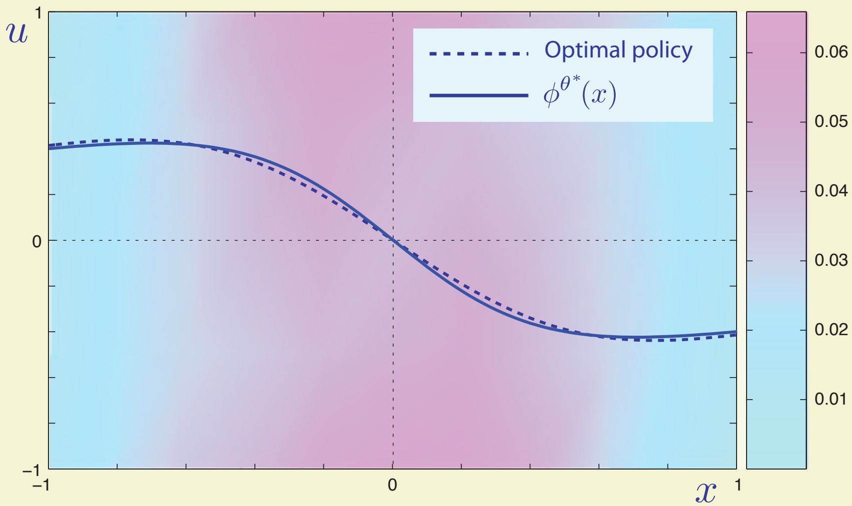


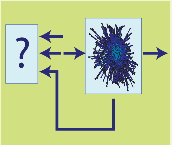
Low amplitude input



High amplitude input

$$u(t) = A(\sin(t) + \sin(\pi t) + \sin(et))$$

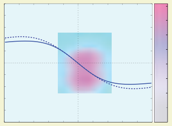
Outline



Coarse models - what to do with them?

Step 1: Recognize
Step 2: Find a stab...
Step 3: Optimality
Step 4: Adjoint
Step 5: Interpret

Q-learning for nonlinear state space models



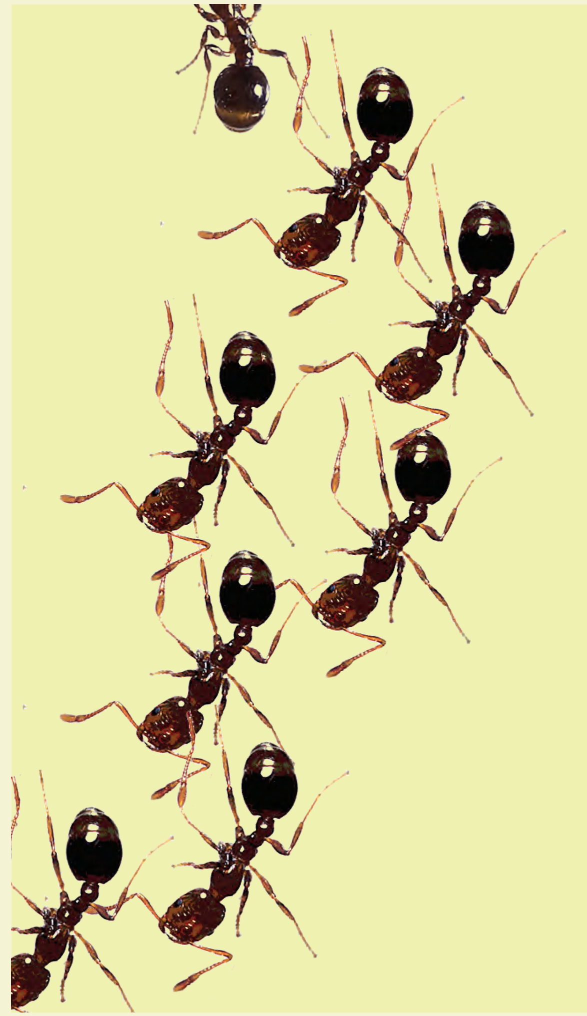
Example: Local approximation

Example: Decentralized control

Multi-agent model

M. Huang, P. E. Caines, and R. P. Malhame. Large-population cost-coupled LQG problems with nonuniform agents: Individual-mass behavior and decentralized ε-Nash equilibria. *IEEE Trans. Auto. Control*, 52(9):1560–1571, 2007.

Huang et. al. Local optimization for global coordination



Multi-agent model

Model: Linear autonomous models - global cost objective

HJB: Individual state + global average

Basis: Consistent with low dimensional LQG model

Results from five agent model:

Multi-agent model

Model: Linear autonomous models - global cost objective

HJB: Individual state + global average

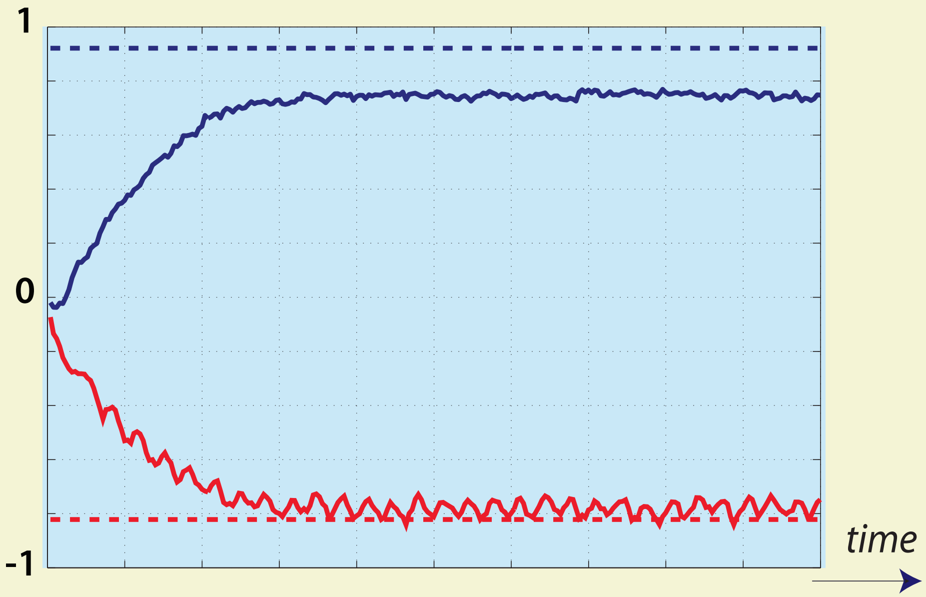
Basis: Consistent with low dimensional LQG model

Results from five agent model:

Estimated state feedback gains

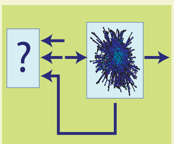
k_x^i *(individual state)*

k_z^i *(ensemble state)*

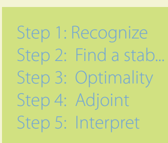


Gains for agent 4: Q-learning sample paths and gains predicted from ∞-agent limit

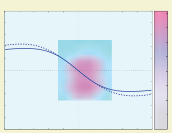
Outline



Coarse models - what to do with them?



Q-learning for nonlinear state space models



Example: Local approximation

Example: Decentralized control

...Conclusions

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Coarse models give tremendous insight

They are also tremendously useful
for design in approximate dynamic programming algorithms

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Current research: Algorithm analysis and improvements
Applications in biology and economics
Analysis of game-theoretic issues
in coupled systems

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