Synchronization of coupled oscillators is a game

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Huibing Yin

Sean P. Meyn

Uday V. Shanbhag

H. Yin, P. G. Mehta, S. P. Meyn and U. V. Shanbhag, “Synchronization of coupled oscillators is a game,” ACC 2010

Acknowledgement

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Minyi Huang

Roland P. Malhame

Peter Caines

M. Huang, P. Caines, and R. Malhame, “Large-population cost-coupled LQG problems with nonuniform agents: Individual-mass behavior and decentralized ε-nash equilibria ,” IEEE TAC, 2007

Millennium bridge

Video of London Millennium bridge from youtube

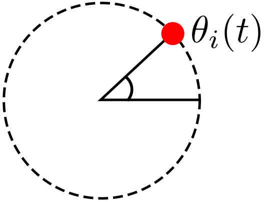
[11] S. H. Strogatz et al., Nature, 2005

Classical Kuramoto model

$$\mathrm{d}\theta_i(t) = \left(\omega_i + \frac{\kappa}{N}\sum_{j=1}^{N}\sin(\theta_j(t)-\theta_i(t))\right)\mathrm{d}t + \sigma\,\mathrm{d}\xi_i(t), \quad i = 1,\ldots,N$$

- ω_i taken from distribution $g(\omega)$ over $[1-\gamma, 1+\gamma]$

  γ — measures the heterogeneity of the population
- κ — measures the strength of coupling



[6] Y. Kuramoto (1975)

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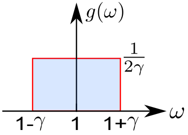

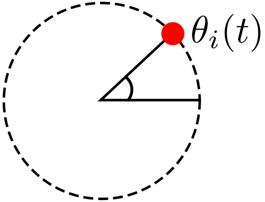

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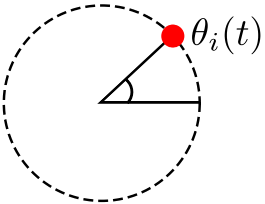

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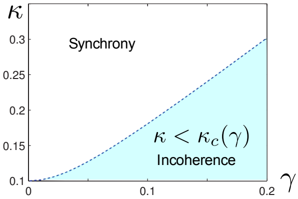

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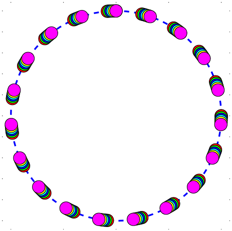
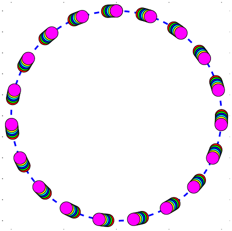
Movies of incoherence and synchrony solution



Incoherence



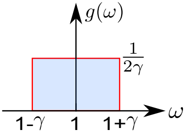
Synchrony

Problem statement

- Dynamics of i^{th} oscillator

$$d\theta_i = (\omega_i + u_i(t))\,dt + \sigma\, d\xi_i, \quad i = 1,\ldots,N, \quad t \geq 0$$

  - $u_i(t)$ — control



- i^{th} oscillator seeks to minimize

$$\eta_i(u_i; u_{-i}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathsf{E}[\,\underbrace{c(\theta_i;\theta_{-i})}_{\text{cost of anarchy}} + \underbrace{\tfrac{1}{2}Ru_i^2}_{\text{cost of control}}\,]\,ds$$

  - $\theta_{-i} = (\theta_j)_{j\neq i}$
  - R — control penalty
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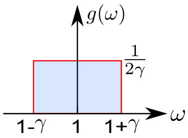

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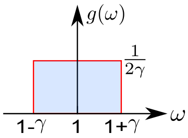

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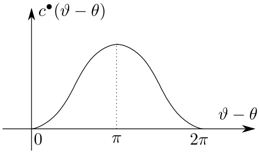

Quiz

In the video you just watched, why were the individuals walking strangely?

A. To show respect to the Queen.
B. Anarchists in the crowd were trying to destabilize the bridge.
C. They were stepping to the beat of the soundtrack "Walk Like an Egyptian."
D. The individuals were trying to maintain their balance.

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"Rational irrationality"

"—behavior that, on the individual level, is perfectly reasonable but that, when aggregated in the marketplace, produces calamity."

- Examples
  - Millennium bridge
  - Financial market

John Cassidy, "Rational Irrationality: The real reason that capitalism is so crash-prone," The New Yorker, 2009

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Hodgkin-Huxley type Neuron model

$$C\frac{\mathrm{d}V}{\mathrm{d}t} = -g_T \cdot m_\infty^2(V) \cdot h \cdot (V - E_T)$$
$$- g_h \cdot r \cdot (V - E_h) - \ldots\ldots$$
$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{h_\infty(V) - h}{\tau_h(V)}$$
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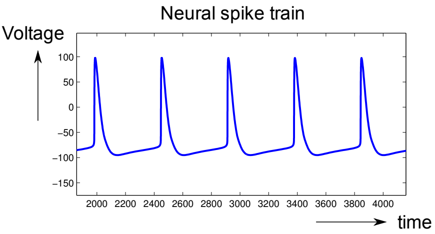

[4] J. Guckenheimer, J. Math. Biol., 1975; [2] J. Moehlis et al., Neural Computation, 2004

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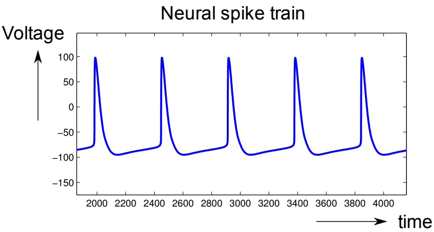

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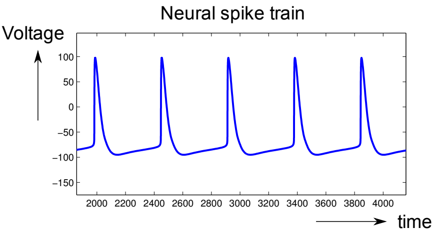

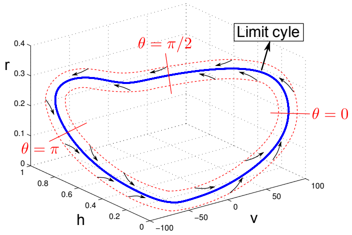

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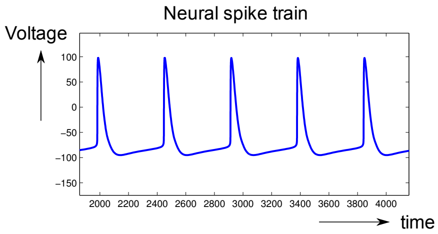
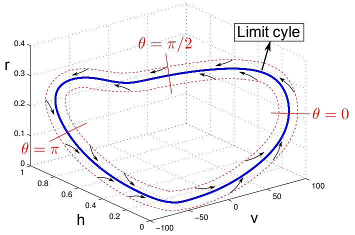
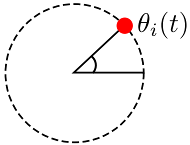




Normal form reduction $\longrightarrow$



$$\dot{\theta}_i = \omega_i + u_i \cdot \Phi(\theta_i)$$

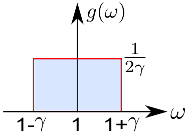
[4] J. Guckenheimer, J. Math. Biol., 1975; [2] J. Moehlis et al., Neural Computation, 2004

Finite oscillator model

- Dynamics of i^{th} oscillator

$$d\theta_i = (\omega_i + u_i(t))\,dt + \sigma\,d\xi_i, \quad i = 1,\dots,N, \quad t \geq 0$$

  - $u_i(t)$ — control



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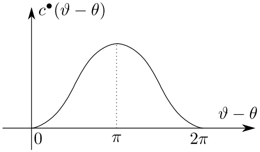
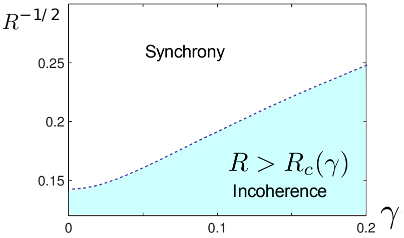
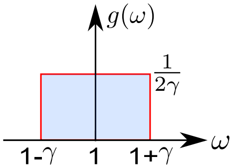


1. Synchronization is a solution of game





$$\mathrm{d}\theta_i = (\omega_i + u_i)\,\mathrm{d}t + \sigma\,\mathrm{d}\xi_i$$

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Yin et al., ACC 2010

Strogatz et al., J. Stat. Phy., 1992

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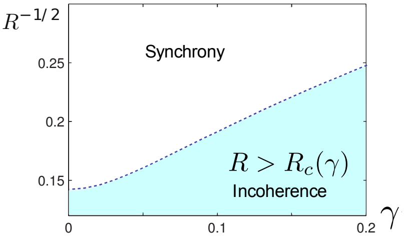

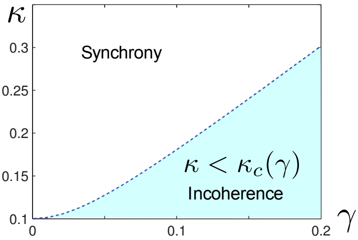

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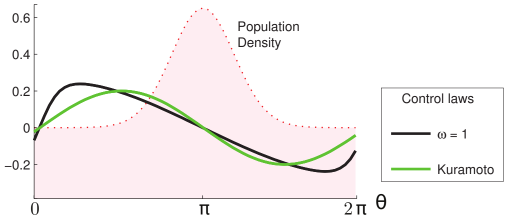
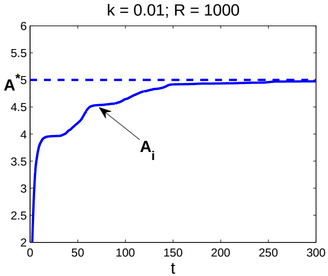
2. Kuramoto control is approximately optimal



$$u_i = -\frac{A_i^*}{R}\frac{1}{N}\sum_{j\neq i}\sin(\theta-\theta_j(t))$$



Learning algorithm:

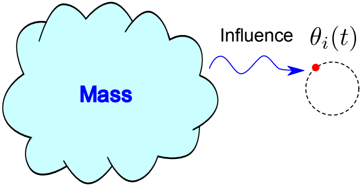
$$\frac{\mathrm{d}A_i}{\mathrm{d}t} = -\varepsilon\ldots$$

Yin et.al. CDC 2010

Overview of model derivation

$$\mathrm{d}\theta_i = (\omega_i + u_i(t))\,\mathrm{d}t + \sigma\,\mathrm{d}\xi_i$$

$$\eta_i(u_i;u_{-i}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathsf{E}[\bar{c}(\theta_i,t) + \tfrac{1}{2}Ru_i^2]\,\mathrm{d}s$$



1. Mean-field approximation
   - Assumption:

$$c(\theta_i;\theta_{-i}(t)) = \frac{1}{N}\sum_{j\neq i} c^{\bullet}(\theta_i,\theta_j) \xrightarrow{N\to\infty} \bar{c}(\theta,t)$$

2. Optimal control of single oscillator
   - Decentralized control structure

[5] M. Huang, P. Caines, and R. Malhame, IEEE TAC, 2007 [HCM]

Single oscillator with given cost

- Dynamics of the oscillator

$$\mathrm{d}\theta_i = (\omega_i + u_i(t))\,\mathrm{d}t + \sigma\,\mathrm{d}\xi_i,\ t \geq 0$$

- The cost function is assumed known

$$\eta_i(u_i;\bar{c}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathsf{E}[\underbrace{\boxed{c(\theta_i;\theta_{-i})}}_{\Uparrow\ \bar{c}(\theta_i(s),s)} + \tfrac{1}{2}Ru_i^2(s)]\,\mathrm{d}s$$

- HJB equation:

$$\partial_t h_i + \omega_i \partial_\theta h_i = \frac{1}{2R}(\partial_\theta h_i)^2 - \bar{c}(\theta,t) + \eta_i^* - \frac{\sigma^2}{2}\partial^2_{\theta\theta} h_i$$

- Optimal control law: $u_i^*(t) = \varphi_i(\theta,t) = -\dfrac{1}{R}\partial_\theta h_i(\theta,t)$

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[1] D. P. Bertsekas (1995); [9] S. P. Meyn, IEEE TAC, 1997

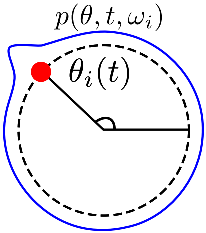
Single oscillator with optimal control

- Dynamics of the oscillator

$$\mathrm{d}\theta_i(t) = \left(\omega_i - \frac{1}{R}\partial_\theta h_i(\theta_i,t)\right)\mathrm{d}t + \sigma\,\mathrm{d}\xi_i(t)$$

- Fokker-Planck equation for pdf $p(\theta,t,\omega_i)$

$$\text{FPK:}\quad \partial_t p + \omega_i \partial_\theta p = \frac{1}{R}\partial_\theta[p(\partial_\theta h)] + \frac{\sigma^2}{2}\partial^2_{\theta\theta}p$$



[7] A. Lasota and M. C. Mackey, “Chaos, Fractals and Noise,” Springer 1994

Mean-field Approximation

- HJB equation for population

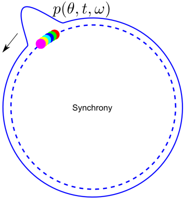
$$\partial_t h + \omega \partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \bar{c}(\theta,t) + \eta(\omega) - \frac{\sigma^2}{2}\partial^2_{\theta\theta} h \qquad \boxed{h(\theta,t,\omega)}$$

- Population density

$$\partial_t p + \omega \partial_\theta p = \frac{1}{R}\partial_\theta [p(\partial_\theta h)] + \frac{\sigma^2}{2}\partial^2_{\theta\theta} p \qquad \boxed{p(\theta,t,\omega)}$$

- Enforce cost consistency

$$\bar{c}(\theta,t) = \int_\Omega \int_0^{2\pi} c^\bullet(\theta,\vartheta) p(\vartheta,t,\omega) g(\omega)\, \mathrm{d}\vartheta\, \mathrm{d}\omega$$
$$\approx \frac{1}{N}\sum_{j\neq i} c^\bullet(\theta,\vartheta)$$



Summary

$$\text{HJB:} \quad \partial_t h + \omega \partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \bar{c}(\theta,t) + \eta^* - \frac{\sigma^2}{2}\partial^2_{\theta\theta} h \quad \Rightarrow \quad h(\theta,t,\omega)$$

$$\text{FPK:} \quad \partial_t p + \omega \partial_\theta p = \frac{1}{R}\partial_\theta [p(\partial_\theta h)] + \frac{\sigma^2}{2}\partial^2_{\theta\theta} p \quad \Rightarrow \quad p(\theta,t,\omega)$$

$$\text{Mean-field approx.:} \quad \bar{c}(\vartheta,t) = \int_\Omega \int_0^{2\pi} c^\bullet(\vartheta,\theta)\, p(\theta,t,\omega)\, g(\omega)\, \mathrm{d}\theta\, \mathrm{d}\omega$$

1. Bellman's optimality principle (H,J,B)
2. Propagation of chaos (F,P,K, Mckean, Vlasov,. . .)
3. Mean-field approximation (Boltzmann, Kac,. . .)
4. Connection to Nash game (Weintraub, HCM, Altman,. . .)

Summary

HJB: $\partial_t h + \omega \partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \boxed{\bar{c}(\theta,t)} + \eta^* - \frac{\sigma^2}{2}\partial^2_{\theta\theta} h \quad \Rightarrow \quad \boxed{h(\theta,t,\omega)}$

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1. Solution of PDE gives ε-Nash equilibrium

- Optimal control law

$$u_i^o = -\frac{1}{R}\partial_\theta h(\theta(t),t,\omega)\Big|_{\omega=\omega_i}$$

- ε-Nash property (as $N \to \infty$)

$$\eta_i(u_i^o; u_{-i}^o) \le \eta_i(u_i; u_{-i}^o) + O\left(\frac{1}{\sqrt{N}}\right), \quad i = 1,\dots,N.$$

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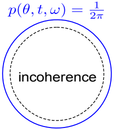
2. Incoherence solution (PDE)

- Incoherence solution

$$h(\theta,t,\omega) = h_0(\theta) := 0 \quad p(\theta,t,\omega) = p_0(\theta) := \frac{1}{2\pi}$$



$$\boxed{h(\theta,t,\omega) = 0} \Rightarrow \quad \underline{\partial_t h} + \omega \underline{\partial_\theta h} = \frac{1}{2R}(\underline{\partial_\theta h})^2 - \bar{c}(\theta,t) + \eta^* - \frac{\sigma^2}{2}\underline{\partial^2_{\theta\theta} h}$$

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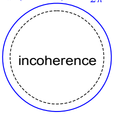

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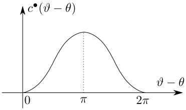

- Assume $c^\bullet(\vartheta,\theta) = c^\bullet(\vartheta-\theta) = \frac{1}{2}\sin^2\left(\frac{\vartheta-\theta}{2}\right)$
- Incoherence solution

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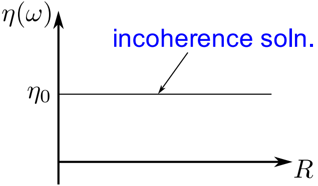
- Optimal control $u = -\frac{1}{R}\partial_\theta h = 0$
- Average cost
  - $\bar{c}(\theta,t) = \int_\Omega \int_0^{2\pi} \frac{1}{2}\sin^2\left(\frac{\theta-\vartheta}{2}\right)\frac{1}{2\pi} g(\omega)\,\mathrm{d}\vartheta\,\mathrm{d}\omega$
  - $\eta^*(\omega) = \bar{c}(\theta,t) = \frac{1}{4} =: \eta_0$ for all $\omega \in \Omega$



$\boxed{\text{No cost of control}}$

2. Incoherence solution (Finite population)

- Closed-loop dynamics $\mathrm{d}\theta_i = (\omega_i + \underbrace{u_i}_{=0})\,\mathrm{d}t + \sigma\,\mathrm{d}\xi_i(t)$

- Average cost

$$\eta_i = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathsf{E}[c(\theta_i;\theta_{-i}) + \underbrace{\tfrac{1}{2}Ru_i^2}_{=0}]\,\mathrm{d}t$$
$$= \lim_{T\to\infty} \frac{1}{N}\sum_{j\neq i}\frac{1}{T}\int_0^T \mathsf{E}[\tfrac{1}{2}\sin^2\left(\frac{\theta_i(t)-\theta_j(t)}{2}\right)]\,\mathrm{d}t$$
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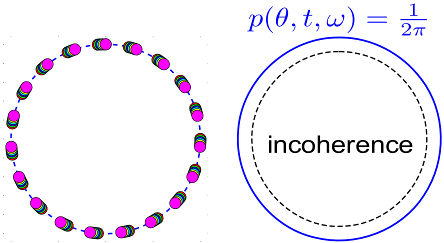

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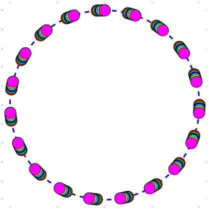
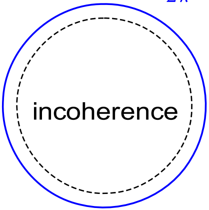


$p(\theta,t,\omega) = \frac{1}{2\pi}$



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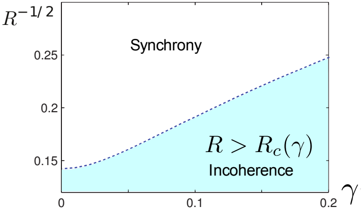
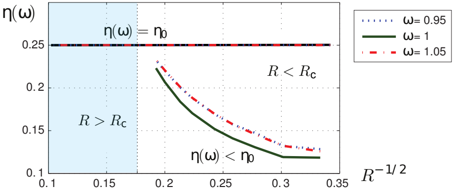
3. Synchronization is a solution of game





$$d\theta_i = (\omega_i + u_i)\,dt + \sigma\, d\xi_i$$

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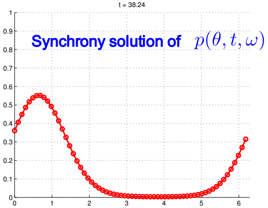

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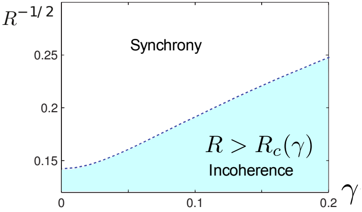

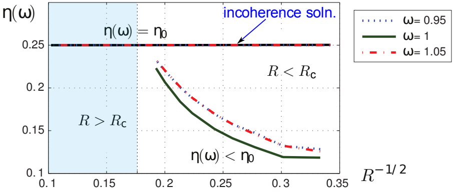

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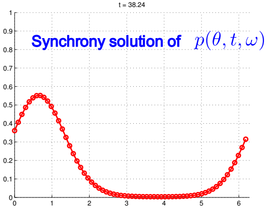

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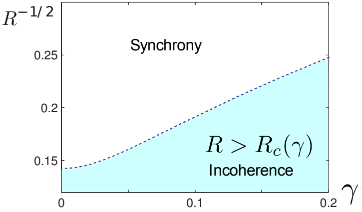

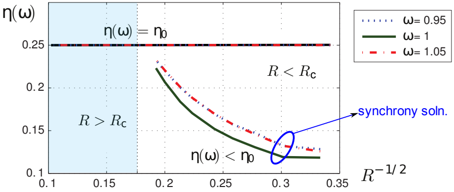

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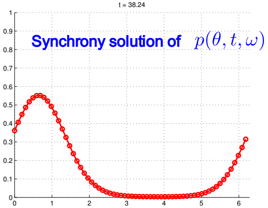

Overview of the steps

$$\text{HJB:}\quad \partial_t h + \omega\partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \boxed{\bar{c}(\theta,t)} + \eta^* - \frac{\sigma^2}{2}\partial^2_{\theta\theta}h \quad \Rightarrow \quad \boxed{h(\theta,t,\omega)}$$

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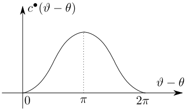

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Linearization and spectra

- Linearized PDE (about incoherence solution)

$$\boxed{\frac{\partial}{\partial t}\tilde{z}(\theta,t,\omega) = \begin{pmatrix} -\omega\partial_\theta \tilde{h} - \bar{c} - \frac{\sigma^2}{2}\partial^2_{\theta\theta}\tilde{h} \\ -\omega\partial_\theta\tilde{p} + \frac{1}{2\pi R}\partial^2_{\theta\theta}\tilde{h} + \frac{\sigma^2}{2}\partial^2_{\theta\theta}\tilde{p} \end{pmatrix} =: \mathscr{L}_R\tilde{z}(\theta,t,\omega)}$$

- Spectrum of the linear operator
  1. Continuous spectrum $\{S^{(k)}\}_{k=-\infty}^{+\infty}$

     $S^{(k)} :=$
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  2. Discrete spectrum

Characteristic eqn: $\displaystyle \frac{1}{8R}\int_\Omega \frac{g(\omega)}{(\lambda - \frac{\sigma^2}{2} + \omega i)(\lambda + \frac{\sigma^2}{2} + \omega i)}\,\mathrm{d}\omega + 1 = 0.$

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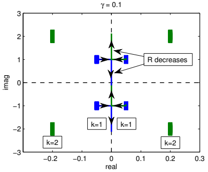

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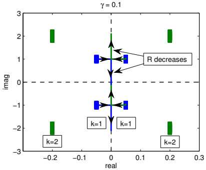

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Bifurcation diagram (Hamiltonian Hopf)

Characteristic eqn: $$\frac{1}{8R}\int_{\Omega}\frac{g(\omega)}{(\lambda-\frac{\sigma^2}{2}+\omega i)(\lambda+\frac{\sigma^2}{2}+\omega i)}\mathrm{d}\omega+1=0.$$

Stability proof

[3] Dellnitz et al., Int. Series Num. Math., 1992

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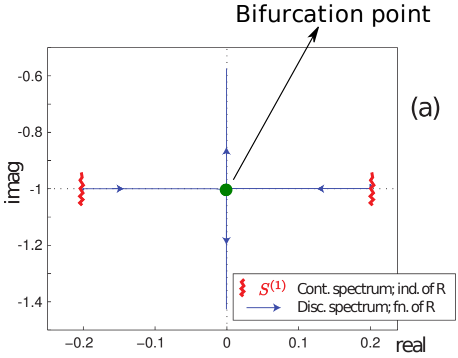

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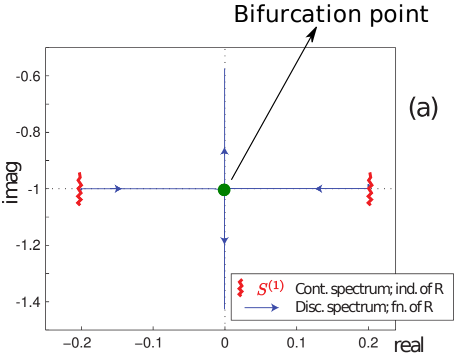

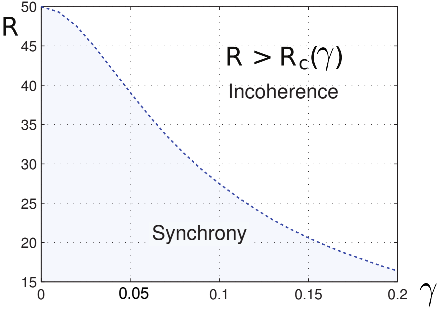

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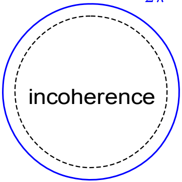
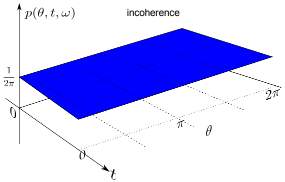
Numerical solution of PDEs

$\boxed{\text{Incoherence}; \quad R = 60}$

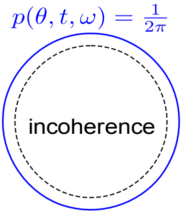
$p(\theta, t, \omega) = \frac{1}{2\pi}$

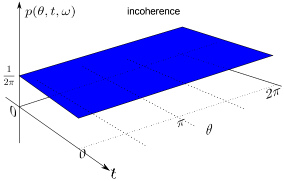




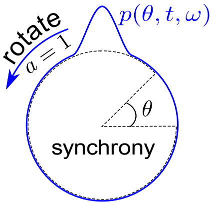
Numerical solution of PDEs

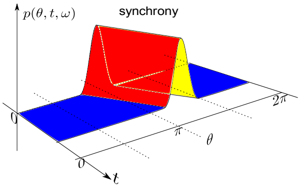
Incoherence; $R = 60$



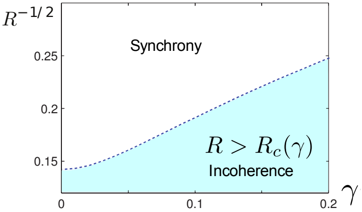


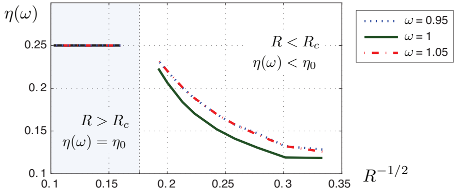
Synchrony; $R = 10$





Bifurcation diagram

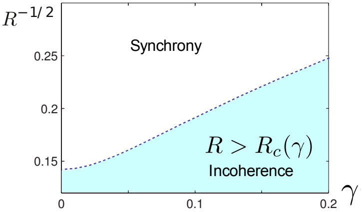


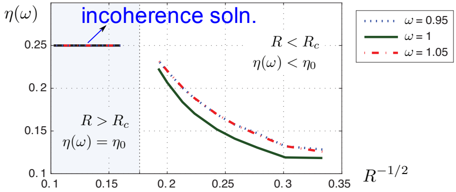


$$\mathrm{d}\theta_i = (\omega_i + u_i)\,\mathrm{d}t + \sigma\,\mathrm{d}\xi_i$$

$$\eta_i(u_i;u_{-i}) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathsf{E}[c(\theta_i;\theta_{-i}) + \tfrac{1}{2}Ru_i^2]\,\mathrm{d}s$$

Bifurcation diagram

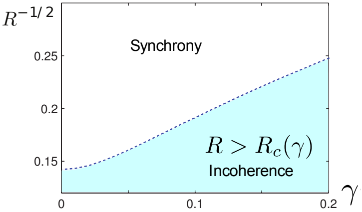


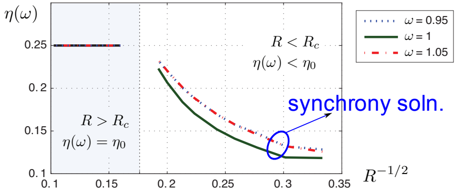


$$\mathrm{d}\theta_i = (\omega_i + u_i)\,\mathrm{d}t + \sigma\,\mathrm{d}\xi_i$$

$$\eta_i(u_i; u_{-i}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathsf{E}[c(\theta_i;\theta_{-i}) + \tfrac{1}{2} R u_i^2]\,\mathrm{d}s$$

Bifurcation diagram



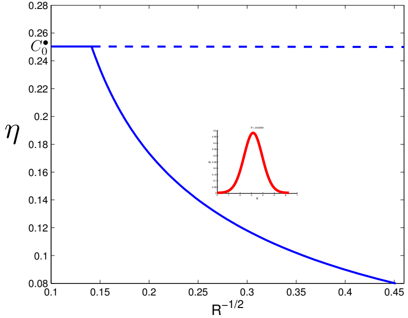


$$d\theta_i = (\omega_i + u_i)\,dt + \sigma\,d\xi_i$$

$$\eta_i(u_i; u_{-i}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathsf{E}[c(\theta_i;\theta_{-i}) + \tfrac{1}{2}Ru_i^2]\,ds$$

Bifurcation diagram with another cost

$$c^{\bullet}(\theta,\vartheta) = 0.25\,(1-\cos(\theta-\vartheta))$$



Bifurcation diagram with another cost

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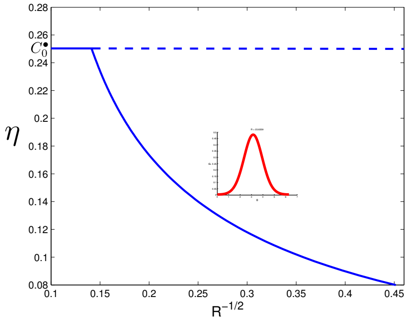
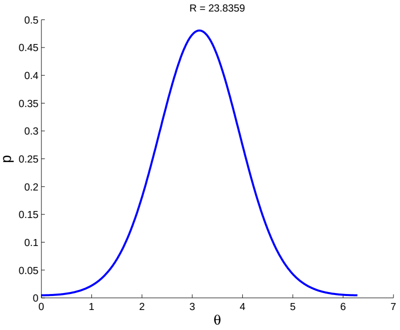
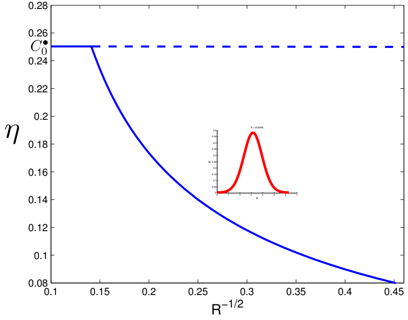
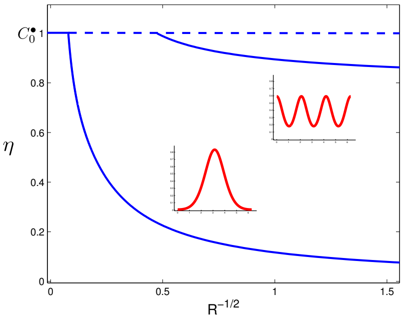




Bifurcation diagram with another cost

$$c^{\bullet}(\theta,\vartheta) = 0.25\,(1-\cos(\theta-\vartheta)) \qquad c^{\bullet}(\theta,\vartheta) = 1-\cos(\theta-\vartheta)-\cos 3(\theta-\vartheta)$$





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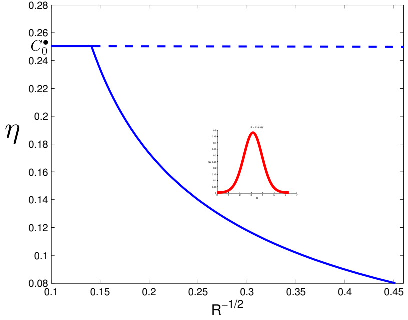
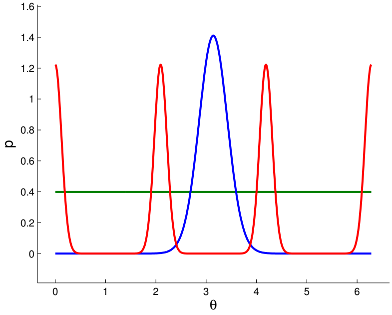
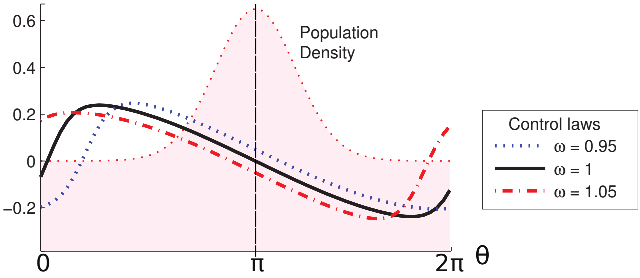




Comparison to Kuramoto law

Control law $u = \varphi(\theta, t, \omega) = -\frac{1}{R}\partial_\theta h(\theta, t, \omega)$



Equivalent control law in Kuramoto oscillator

$$u_i^{(\text{Kur})} = \frac{\kappa}{N}\sum_{j=1}^{N}\sin(\theta_j(t) - \theta_i) \overset{N\to\infty}{\approx} \kappa_0 \sin(\vartheta_0 + t - \theta_i)$$

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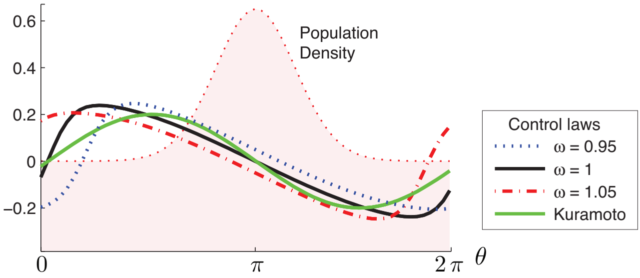

Equivalent control law in Kuramoto oscillator

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- Optimality equation $\min_{u_i}\{\underbrace{c(\theta;\theta_{-i}(t)) + \frac{1}{2}Ru_i^2 + \mathscr{D}_{u_i}h_i(\theta,t)}_{=:\,H_i(\theta,u_i;\theta_{-i}(t))}\} = \eta_i^*$

- Optimal control law Kuramoto law

$$u_i^* = -\frac{1}{R}\partial_\theta h_i(\theta,t) \qquad u_i^{(\text{Kur})} = -\frac{\kappa}{N}\sum_{j\neq i}\sin(\theta_i - \theta_j(t))$$

- Parameterization:

$$H_i^{(A_i,\phi_i)}(\theta,u_i;\theta_{-i}(t)) = c(\theta;\theta_{-i}(t)) + \tfrac{1}{2}Ru_i^2 + (\omega_i - 1 + u_i)A_iS^{(\phi_i)} + \frac{\sigma^2}{2}A_iC^{(\phi_i)}$$

where

$$S^{(\phi)}(\theta,\theta_{-i}) = \frac{1}{N}\sum_{j\neq i}\sin(\theta - \theta_j - \phi), \quad C^{(\phi)}(\theta,\theta_{-i}) = \frac{1}{N}\sum_{j\neq i}\cos(\theta - \theta_j - \phi)$$

- Approx. optimal control:

$$u_i^{(A_i,\phi_i)} = \underset{u_i}{\operatorname{argmin}}\{H_i^{(A_i,\phi_i)}(\theta,u_i;\theta_{-i}(t))\} = -\frac{A_i}{R}S^{(\phi_i)}(\theta,\theta_{-i})$$

- Optimality equation $$\min_{u_i}\{\underbrace{c(\theta;\theta_{-i}(t))+\tfrac{1}{2}Ru_i^2+\mathscr{D}_{u_i}h_i(\theta,t)}_{=:\,H_i(\theta,u_i;\theta_{-i}(t))}\}=\eta_i^*$$
- Optimal control law $$u_i^*=-\frac{1}{R}\partial_\theta h_i(\theta,t)$$

  Kuramoto law $$u_i^{(\text{Kur})}=-\frac{\kappa}{N}\sum_{j\neq i}\sin(\theta_i-\theta_j(t))$$
- Parameterization:

$$H_i^{(A_i,\phi_i)}(\theta,u_i;\theta_{-i}(t))=c(\theta;\theta_{-i}(t))+\tfrac{1}{2}Ru_i^2+(\omega_i-1+u_i)A_iS^{(\phi_i)}+\frac{\sigma^2}{2}A_iC^{(\phi_i)}$$

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- Approx. optimal control:

$$u_i^{(A_i,\phi_i)}=\operatorname*{argmin}_{u_i}\{H_i^{(A_i,\phi_i)}(\theta,u_i;\theta_{-i}(t))\}=-\frac{A_i}{R}S^{(\phi_i)}(\theta,\theta_{-i})$$

- Optimality equation $\min_{u_i}\{\underbrace{c(\theta;\theta_{-i}(t))+\tfrac{1}{2}Ru_i^2+\mathscr{D}_{u_i}h_i(\theta,t)}_{=:\,H_i(\theta,u_i;\theta_{-i}(t))}\}=\eta_i^*$

- Optimal control law

$$u_i^* = -\frac{1}{R}\partial_\theta h_i(\theta,t)$$

Kuramoto law

$$u_i^{(\text{Kur})} = -\frac{\kappa}{N}\sum_{j\neq i}\sin(\theta_i-\theta_j(t))$$

- Parameterization:

$$H_i^{(A_i,\phi_i)}(\theta,u_i;\theta_{-i}(t)) = c(\theta;\theta_{-i}(t))+\tfrac{1}{2}Ru_i^2+(\omega_i-1+u_i)A_iS^{(\phi_i)}+\frac{\sigma^2}{2}A_iC^{(\phi_i)}$$

where

$$S^{(\phi)}(\theta,\theta_{-i}) = \frac{1}{N}\sum_{j\neq i}\sin(\theta-\theta_j-\phi), \quad C^{(\phi)}(\theta,\theta_{-i}) = \frac{1}{N}\sum_{j\neq i}\cos(\theta-\theta_j-\phi)$$

- Approx. optimal control:

$$u_i^{(A_i,\phi_i)} = \operatorname*{argmin}_{u_i}\{H_i^{(A_i,\phi_i)}(\theta,u_i;\theta_{-i}(t))\} = -\frac{A_i}{R}S^{(\phi_i)}(\theta,\theta_{-i})$$

- Optimality equation $$\min_{u_i}\{\underbrace{c(\theta;\theta_{-i}(t)) + \tfrac{1}{2}Ru_i^2 + \mathscr{D}_{u_i}h_i(\theta,t)}_{=:\,H_i(\theta,u_i;\theta_{-i}(t))}\} = \eta_i^*$$
- Optimal control law

$$u_i^* = -\frac{1}{R}\partial_\theta h_i(\theta,t)$$

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- Parameterization:

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- Approx. optimal control:

$$u_i^{(A_i,\phi_i)} = \operatorname*{argmin}_{u_i}\{H_i^{(A_i,\phi_i)}(\theta,u_i;\theta_{-i}(t))\} = -\frac{A_i}{R}S^{(\phi_i)}(\theta,\theta_{-i})$$

- Bellman error:

Pointwise: $$\mathscr{L}^{(A_i,\phi_i)}(\theta,t) = \min_{u_i}\{H_i^{(A_i,\phi_i)}\} - \eta_i^{(A_i^*,\phi_i^*)}$$

- Simple gradient descent algorithm

$$\tilde{e}(A_i,\phi_i) = \sum_{k=1}^{2} |\langle \mathscr{L}^{(A_i,\phi_i)}, \tilde{\varphi}_k(\theta)\rangle|^2$$

$$\frac{\mathrm{d}A_i}{\mathrm{d}t} = -\varepsilon\frac{\mathrm{d}\tilde{e}(A_i,\phi_i)}{\mathrm{d}A_i}, \qquad \frac{\mathrm{d}\phi_i}{\mathrm{d}t} = -\varepsilon\frac{\mathrm{d}\tilde{e}(A_i,\phi_i)}{\mathrm{d}\phi_i} \qquad (*)$$

Theorem (Convergence)

Assume population is in synchrony. The i^{th} oscillator updates according to $(*)$. Then

$$A_i(t) \to A^* = \frac{1}{2\sigma^2}$$

- *The pointwise Bellman error* $\mathscr{L}^{(A_i,0)}(\theta,t) = \varepsilon(R)\cos 2(\theta - t)$ *where* $\varepsilon(R) = \dfrac{1}{16R\sigma^4}$

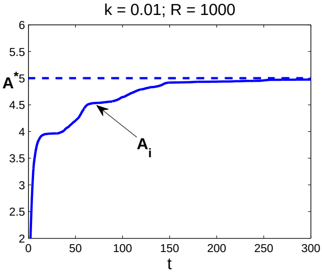
Phase transition

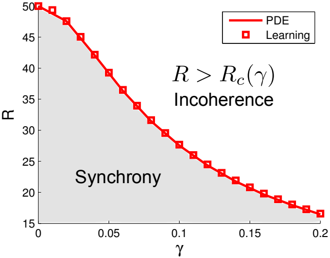
Suppose all oscillators use approx. optimal control law:

$$u_i = -\frac{A^*}{R}\frac{1}{N}\sum_{j\neq i}\sin(\theta_i - \theta_j(t))$$

then the phase transition boundary is

$$R_c(\gamma) = \begin{cases} \frac{1}{2\sigma^4} & \text{if } \gamma = 0 \\ \frac{1}{4\sigma^2\gamma}\tan^{-1}\left(\frac{2\gamma}{\sigma^2}\right) & \text{if } \gamma > 0 \end{cases}$$

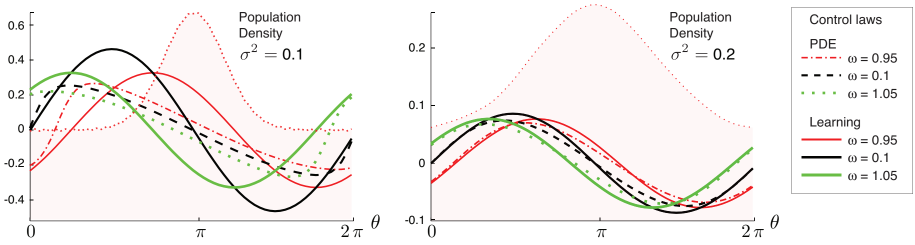




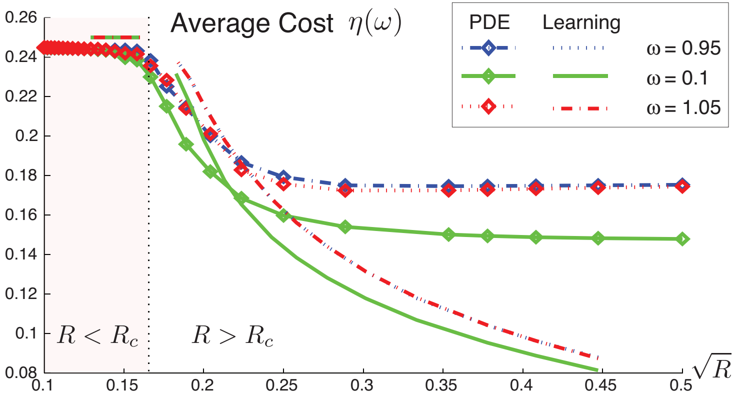
Comparison of control laws

From PDE: $u_i = \varphi(\theta,t,\omega)\Big|_{\omega=\omega_i} = -\frac{1}{R}\partial_\theta h(\theta,t,\omega)\Big|_{\omega=\omega_i}$

From learning: $u_i = -\frac{A^*}{R}\frac{1}{N}\sum_{j\neq i}\sin(\theta_i - \theta_j(t) - \phi_i)$



Comparison of average cost



Thank you!

Website: http://www.mechse.illinois.edu/research/mehtapg

Huibing Yin

Sean P. Meyn

Uday V. Shanbhag

H. Yin, P. G. Mehta, S. P. Meyn and U. V. Shanbhag, “Synchronization of coupled oscillators is a game,” ACC 2010

Part I

Apendix

Infinit population limit

- Recall the PDE model

$$\partial_t h + \omega \partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \bar{c}(\theta,t) + \eta^* - \frac{\sigma^2}{2}\partial^2_{\theta\theta} h$$

$$\partial_t p + \omega \partial_\theta p = \frac{1}{R}\partial_\theta[p(\partial_\theta h)] + \frac{\sigma^2}{2}\partial^2_{\theta\theta} p$$

$$\bar{c}(\vartheta,t) = \int_\Omega \int_0^{2\pi} c^\bullet(\vartheta,\theta)p(\theta,t,\omega)g(\omega)\,\mathrm{d}\theta\,\mathrm{d}\omega$$

  - Solution $h(\theta,t,\omega)$
  - Control $u_i^o(t) := -\frac{1}{R}\partial_\theta h(\theta,t,\omega)\big|_{\omega=\omega_i}$
  - By construction:

$$\eta_i(u_i^o;\bar{c}) \le \eta_i(u_i;\bar{c})$$

Finite population

- Finite population cost

$$\bar{c}_i^{(N)}(\vartheta,t) := \mathsf{E}\left[\frac{1}{N}\sum_{j\neq i} c^{\bullet}\big(\theta_i(t);\theta^o_{-i}(t)\big|\theta_i(t)=\vartheta\big)\right]$$

$\theta^o_{-i} = (\theta^o_1,\ldots,\theta^o_{i-1},\theta^o_{i+1},\ldots,\theta^o_N)$ with $\theta^o_i(t)$ obtained from dynamics with control $u_i = u^o_i(t)$

- Lemma 1: for each $i = 1,\ldots,N$

$$\max_{\vartheta\in[0,2\pi]}\left[\limsup_{T\to\infty}\frac{1}{T}\int_0^T |\bar{c}(\vartheta,s)-\bar{c}_i^{(N)}(\vartheta,s)|\,\mathrm{d}s\right] = O\Big(\frac{1}{\sqrt{N}}\Big)$$

ε-Nash equilibrium

Theorem

The control $\boxed{\{u_i^o(t) = -\frac{1}{R}\partial_\theta h(\theta(t),t,\omega)\big|_{\omega=\omega_i}\}_{i=1}^N}$ *is an ε-Nash equilibrium with respect to the finite population cost*

$$\eta_i^{(POP)}(u_i;u_{-i}^o) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \bar{c}_i^{(N)} + \frac{1}{2}Ru_i^2\,\mathrm{d}s;$$

i.e., for any adapted control u_i,

$$\eta_i^{(POP)}(u_i^o;u_{-i}^o) \leq \eta_i^{(POP)}(u_i;u_{-i}^o) + O(\frac{1}{\sqrt{N}}), \quad i=1,\ldots,N.$$

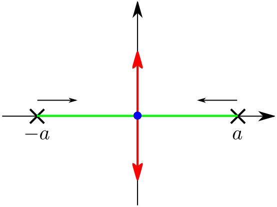
$$\boxed{\bar{c}_i^{(N)}(\vartheta,t) := \mathsf{E}[c(\theta_i(t);\theta_{-i}^o(t))|\theta_i(t)=\vartheta]}$$

Stability proof

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ax + by, \; a > 0 \qquad \Leftarrow \text{FPK eqn}$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = cx + ay, \; bc < 0 \qquad \Leftarrow \text{HJB eqn}$$

with boundary conditions $x(0) = x_0$ and $y(\infty) = 0$

$$x(t) = e^{at}x_0 + b\int_0^t e^{-a(t-\tau)}y(\tau)\,\mathrm{d}\tau,$$
$$y(t) = -c\int_t^\infty e^{a(t-\tau)}x(\tau)\,\mathrm{d}\tau.$$



- For $x_0 = 0$, equilibrium solution $x(t) \equiv 0, \; y(t) \equiv 0$
- Characteristic equation $\sigma^2 = a^2 + bc$.
- Stable in the sense that

The equilibrium solution is *asymptotically stable* if for any initial perturbation x_0, the solution $x(t) \to 0$ as $t \to \infty$.

Return to spectrum

Algorithm for solving PDEs

HJB: $$\partial_t h + \omega \partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \bar{c}(\theta,t) + \eta^* - \frac{\sigma^2}{2}\partial^2_{\theta\theta} h \quad \Rightarrow \quad h(\theta,t,\omega)$$

FPK: $$\partial_t p + \omega \partial_\theta p = \frac{1}{R}\partial_\theta [p(\partial_\theta h)] + \frac{\sigma^2}{2}\partial^2_{\theta\theta} p \quad \Rightarrow \quad p(\theta,t,\omega)$$

$$\bar{c}(\vartheta,t) = \int_\Omega \int_0^{2\pi} c^\bullet(\vartheta,\theta)\, p(\theta,t,\omega)\, g(\omega)\,\mathrm{d}\theta\,\mathrm{d}\omega$$

$V(\theta,t,\omega) = h(\theta,T-s,\omega)$

Iterative algorithm

$$\partial_s V = \omega\partial_\theta V - \frac{1}{2R}(\partial_\theta V)^2 + \frac{1}{2}\hat{\phi}(\theta,T-s) + \frac{\sigma^2}{2}\partial^2_{\theta\theta} V$$

s T Backward time 0

$\bar{c}(\theta,T-s)$

$u(\theta,t) = -\frac{1}{R}\partial_\theta V(\theta,s)$

$$\partial_t p = -\,\omega\partial_\theta p - \partial_\theta[up] + \frac{\sigma^2}{2}\partial^2_{\theta\theta} p$$

0 Forward time T t

Algorithm for solving PDEs

HJB: $$\partial_t h + \omega \partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \bar{c}(\theta,t) + \eta^* - \frac{\sigma^2}{2}\partial^2_{\theta\theta} h \quad \Rightarrow \quad h(\theta,t,\omega)$$

FPK: $$\partial_t p + \omega \partial_\theta p = \frac{1}{R}\partial_\theta [p(\partial_\theta h)] + \frac{\sigma^2}{2}\partial^2_{\theta\theta} p \quad \Rightarrow \quad p(\theta,t,\omega)$$

$$\bar{c}(\vartheta,t) = \int_\Omega \int_0^{2\pi} c^\bullet(\vartheta,\theta)\, p(\theta,t,\omega)\, g(\omega)\,\mathrm{d}\theta\,\mathrm{d}\omega$$

$V(\theta,t,\omega) = h(\theta,T-s,\omega)$

Iterative algorithm

$$\partial_s V = \omega \partial_\theta V - \frac{1}{2R}(\partial_\theta V)^2 + \frac{1}{2}\hat{\phi}(\theta,T-s) + \frac{\sigma^2}{2}\partial^2_{\theta\theta} V$$

s T Backward time 0

$\bar{c}(\theta,T-s)$

$u(\theta,t) = -\frac{1}{R}\partial_\theta V(\theta,s)$

$$\partial_t p = -\omega \partial_\theta p - \partial_\theta [up] + \frac{\sigma^2}{2}\partial^2_{\theta\theta} p$$

0 Forward time T t

Algorithm for solving PDEs

HJB: $\partial_t h + \omega \partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \boxed{\bar{c}(\theta,t)} + \eta^* - \frac{\sigma^2}{2}\partial^2_{\theta\theta} h \quad \Rightarrow \quad \boxed{h(\theta,t,\omega)}$

FPK: $\partial_t p + \omega \partial_\theta p = \frac{1}{R}\partial_\theta[p(\boxed{\partial_\theta h})] + \frac{\sigma^2}{2}\partial^2_{\theta\theta} p \quad \Rightarrow \quad \boxed{p(\theta,t,\omega)}$

$$\boxed{\bar{c}(\vartheta,t)} = \int_\Omega \int_0^{2\pi} c^\bullet(\vartheta,\theta)\, \boxed{p(\theta,t,\omega)}\, g(\omega)\, \mathrm{d}\theta\, \mathrm{d}\omega$$

$V(\theta,t,\omega) = h(\theta,T-s,\omega)$

Iterative algorithm

$$\partial_s V = \omega\partial_\theta V - \frac{1}{2R}(\partial_\theta V)^2 + \frac{1}{2}\hat{\phi}(\theta,T-s) + \frac{\sigma^2}{2}\partial^2_{\theta\theta} V$$

s T Backward time 0

$\bar{c}(\theta, T-s)$

$u(\theta,t) = -\frac{1}{R}\partial_\theta V(\theta,s)$

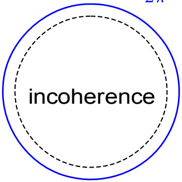
$$\partial_t p = -\omega\partial_\theta p - \partial_\theta[up] + \frac{\sigma^2}{2}\partial^2_{\theta\theta} p$$

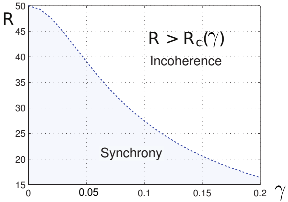
0 Forward time T t

Numerical solution of PDEs

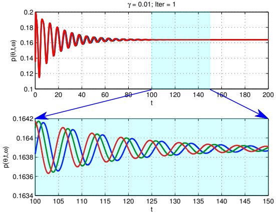
Incoherence; $\quad R = 60$

$$p(\theta, t, \omega) = \frac{1}{2\pi}$$

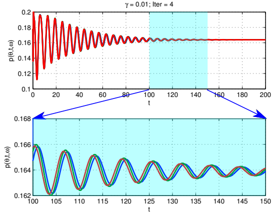




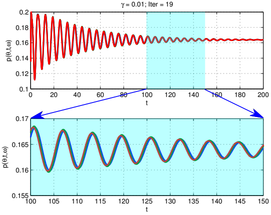
$p(\theta, t, \omega)$ at given position θ



Iter = 1



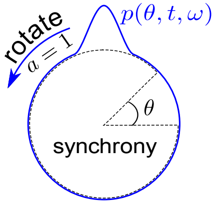
Iter=4

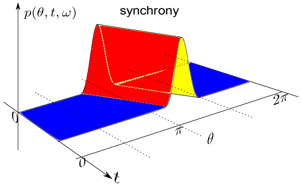


Iter=19

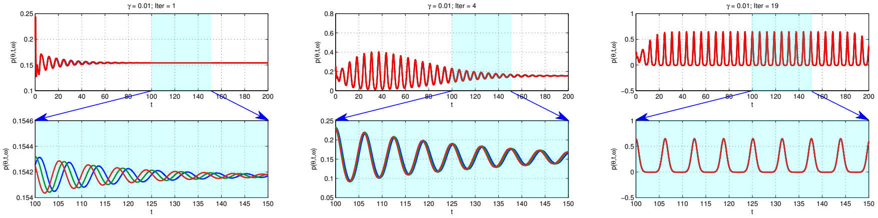
Numerical solution of PDEs

Synchrony; $R = 10$





$p(\theta, t, \omega)$ at given position θ



Iter = 1 Iter=4 Iter=19

Linearization

$$\frac{\partial}{\partial t}\tilde{z}(\theta,t,\omega) = \begin{pmatrix} -\omega\partial_\theta\tilde{h} - \bar{c} - \frac{\sigma^2}{2}\partial^2_{\theta\theta}\tilde{h} \\ -\omega\partial_\theta\tilde{p} + \frac{1}{2\pi R}\partial^2_{\theta\theta}\tilde{h} + \frac{\sigma^2}{2}\partial^2_{\theta\theta}\tilde{p} \end{pmatrix}$$

- Eigenvector problem $\lambda Z = \mathscr{L}_R Z$
- Fourier series expansion with respect to θ

$$H = \sum_{k=-\infty}^{+\infty} H_k(\omega)e^{ik\theta}, \quad P = \sum_{k=-\infty}^{+\infty} P_k(\omega)e^{ik\theta}$$

- Decomposition of the linear operator $\mathscr{L}_R = \oplus_k \mathscr{L}_R^{(k)}$

Linearization (cont.)

- Individual operator

$$\mathscr{L}_R^{(1)} := \begin{pmatrix} \frac{\sigma^2}{2} - \omega i & \frac{\pi}{4}\int_\Omega \cdot g(\omega)\,\mathrm{d}\omega \\ -\frac{1}{2\pi R} & -\frac{\sigma^2}{2} - \omega i \end{pmatrix}$$

$$\mathscr{L}_R^{(k)} := \begin{pmatrix} \frac{\sigma^2}{2}k^2 - k\omega i & 0 \\ -\frac{k^2}{2\pi R} & -\frac{\sigma^2}{2}k^2 - k\omega i \end{pmatrix}, \quad k \geq 2$$

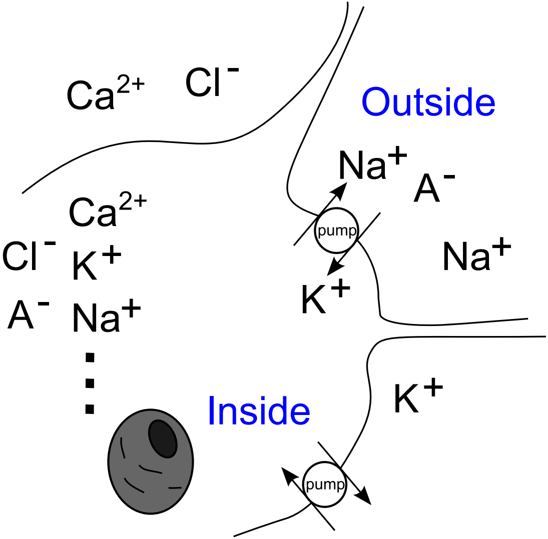
$$\mathscr{L}_R^{(-k)} = \overline{\mathscr{L}_R^{(k)}}$$

- Continuous spectrum $\{S^{(k)}\}_{k=-\infty}^{+\infty}$

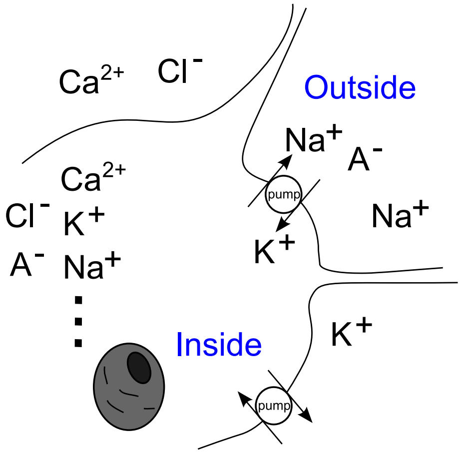
$$S^{(k)} := \left\{ \lambda \in \mathbb{C} \middle| \lambda = \pm\frac{\sigma^2}{2}k^2 - k\omega i \text{ for all } \omega \in \Omega \right\}$$

- Discrete spectrum coincides with the discrete spectrum of $\mathscr{L}_R^{(\pm 1)}$

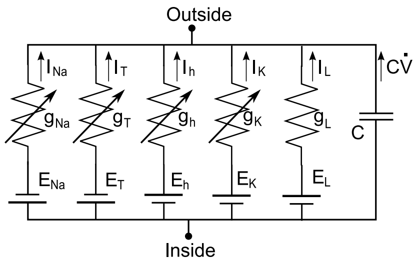
Neuron



Neuron



Equivalent circuits

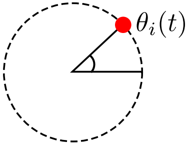


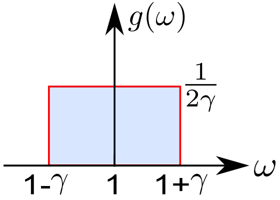
Finite oscillator model

- Dynamics of i^{th} oscillator [10]:

$$\mathrm{d}\theta_i = (\omega_i + u_i(t))\,\mathrm{d}t + \sigma\,\mathrm{d}\xi_i, \quad i = 1, \ldots, N, \quad t \geq 0 \tag{1}$$

  - θ_i— phase of the oscillator
  - $\{\xi_i\}$ — mutually independent Wiener processes
  - ω_i — constant and chosen independently according to a fixed distribution with density g
  - $u_i(t)$ — control





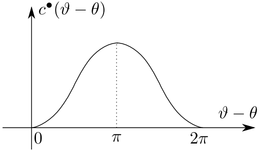
Finite oscillator model (cont.)

- i^{th} oscillator seeks to minimize

$$\eta_i^{(\text{POP})}(u_i;u_{-i}) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathsf{E}[\ \underbrace{c(\theta_i;\theta_{-i})}_{\text{cost of anarchy}} + \underbrace{\tfrac{1}{2}Ru_i^2}_{\text{cost of control}}\ ]\,\mathrm{d}s$$

  - $\theta_{-i} = (\theta_j)_{j\neq i}$
  - R — control penalty
  - $c(\cdot)$ — cost function

$$c(\theta_i;\theta_{-i}) = \frac{1}{N}\sum_{j\neq i} c^\bullet(\theta_i,\theta_j), \quad c^\bullet \geq 0$$



- $\{u_i^*\}_{i=1}^N$ Nash equilibrium if

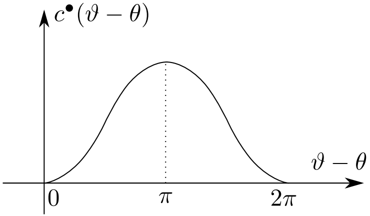
$$u_i^* \text{ minimizes } \eta_i^{(\text{POP})}(u_i;u_{-i}^*), \text{ for } i = 1,\ldots,N.$$

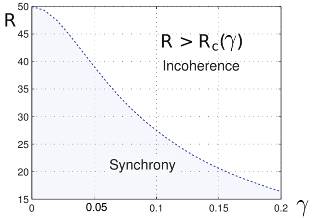
Intuition

- i^{th} oscillator seeks to minimize

$$\eta_i^{(\text{POP})}(u_i;u_{-i}) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathsf{E}[\ \underbrace{c(\theta_i;\theta_{-i})}_{\text{cost of anarchy}} + \underbrace{\tfrac{1}{2}Ru_i^2}_{\text{cost of control}}\ ]\,\mathrm{d}s$$

- Trade-off between reducing the two costs
  - Cost associated with $\theta_i \neq \theta_j$: $c(\theta_i;\theta_{-i}) = \dfrac{1}{N}\sum_{j\neq i} c^{\bullet}(\theta_i - \theta_j)$
  - Cost associated with control: $\frac{1}{2}Ru_i^2$
- Qualitative macro-behavior change when R varies





PDE model (as $N \longrightarrow \infty$)

- HJB equation $\Rightarrow$ $\boxed{\text{obtain optimal control}}$

$$\partial_t h + \omega \partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \bar{c}(\theta,t) + \eta^* - \frac{\sigma^2}{2}\partial^2_{\theta\theta} h \tag{2}$$

  - $h(\theta,t,\omega)$— relative value function
  - $\eta^*(\omega)$— average optimal cost
  - Optimal feed-back control law $\varphi(\theta,t,\omega) := -\frac{1}{R}\partial_\theta h(\theta,t,\omega)$

PDE model (as $N \to \infty$)

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- Fokker-Planck-Kolmogorov (FPK) equation

$$\mathrm{d}\theta_i(t) = (\omega_i + u_i(t))\,\mathrm{d}t + \sigma\,\mathrm{d}\xi_i,\; i = 1,\ldots,N$$

replaced by: $\partial_t p + \partial_\theta\Big((\omega + \varphi)p\Big) = \frac{\sigma^2}{2}\partial^2_{\theta\theta} p$

  - $p(\theta,t,\omega)$ — probability density function

[10] Strogatz et al., Journal of Statistical Physics, 1991

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PDE model (as $N \to \infty$)

- HJB equation $\Rightarrow$ obtain optimal control

$$\partial_t h + \omega \partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \bar{c}(\theta, t) + \eta^* - \frac{\sigma^2}{2}\partial^2_{\theta\theta} h \tag{2}$$

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  - $\eta^*(\omega)$— average optimal cost
  - Optimal feed-back control law $\varphi(\theta, t, \omega) := -\frac{1}{R}\partial_\theta h(\theta, t, \omega)$

- FPK equation $\Rightarrow$ evolution of population phase angles

$$\partial_t p + \omega \partial_\theta p = \frac{1}{R}\partial_\theta [p(\partial_\theta h)] + \frac{\sigma^2}{2}\partial^2_{\theta\theta} p \tag{3}$$

PDE model (as $N \to \infty$)

- HJB equation $\Rightarrow$ $\boxed{\text{obtain optimal control}}$

$$\partial_t h + \omega \partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \bar{c}(\theta, t) + \eta^* - \frac{\sigma^2}{2}\partial^2_{\theta\theta} h \qquad (2)$$

  - $h(\theta, t, \omega)$— relative value function
  - $\eta^*(\omega)$— average optimal cost
  - Optimal feed-back control law $\varphi(\theta, t, \omega) := -\frac{1}{R}\partial_\theta h(\theta, t, \omega)$

- FPK equation $\Rightarrow$ $\boxed{\text{evolution of population phase angles}}$

$$\partial_t p + \omega \partial_\theta p = \frac{1}{R}\partial_\theta[p(\partial_\theta h)] + \frac{\sigma^2}{2}\partial^2_{\theta\theta} p \quad \Rightarrow \qquad (3)$$

Generate cost: $\int_\Omega \int_0^{2\pi} c^\bullet(\vartheta, \theta) p(\theta, t, \omega) g(\omega)\, \mathrm{d}\theta\, \mathrm{d}\omega$

Consistency

- Consistency requirement on deterministic mass influence

$$\bar{c}(\vartheta, t) = \int_{\Omega} \int_{0}^{2\pi} c^{\bullet}(\vartheta, \theta) p(\theta, t, \omega) g(\omega) \, \mathrm{d}\theta \, \mathrm{d}\omega$$

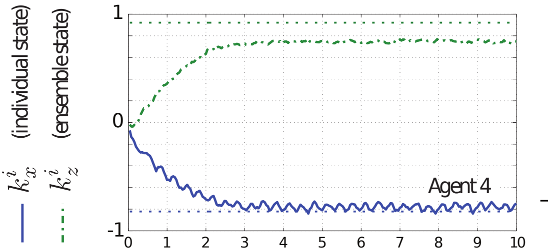
Recall: $$c(\theta_i; \theta_{-i}) = \frac{1}{N} \sum_{j \neq i} c^{\bullet}(\theta_i, \theta_j)$$

Q-learning

- Dynamics of i^{th} agent $\frac{\mathrm{d}}{\mathrm{d}t}x_i = a_i x_i + b_i u_i$
- i^{th} agent seeks to minimize

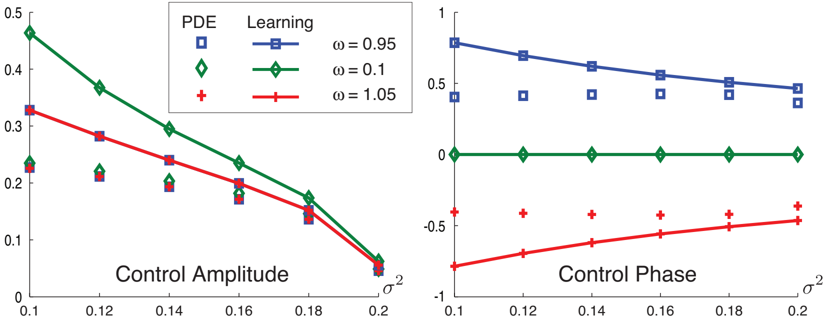
$$\eta_i = \lim_{T\to\infty} \int_0^T e^{-\gamma s} \left((x_i(s) - z(s))^2 + u_i^2(s) \right) \mathrm{d}s$$

- Control $u_i = -k_x^i x_i - k_z^i z$



[8] P. Mehta and S. Meyn, CDC, 2009

Control comparison



Rational irrationality

The Prisoner's Dilemma is the obverse of Adam Smith's theory of the invisible hand, in which the free market coördinates the behavior of self-seeking individuals to the benefit of all. Each businessman "intends only his own gain," Smith wrote in "The Wealth of Nations," "and he is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention." But in a market environment the individual pursuit of self-interest, however rational, can give way to collective disaster. The invisible hand becomes a fist.

John Cassidy, "Rational Irrationality: The real reason that capitalism is so crash-prone," The New Yorker, 2009

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