Chapter 1
Dynamic Competitive Equilibria in Electricity Markets

Gui Wang, Matias Negrete-Pincetic, Anupama Kowli, Ehsan Shafieepoorfard, Sean Meyn and Uday V. Shanbhag

Abstract This chapter addresses the economic theory of electricity markets, viewed from an idealized competitive equilibrium setting, taking into account volatility and the physical and operational constraints inherent to transmission and generation. In a general dynamic setting, we establish many of the standard conclusions of competitive equilibrium theory: Market equilibria are efficient, and average prices coincide with average marginal costs. However, *these conclusions hold only on average.* An important contribution of this chapter is the explanation of the exotic behavior of electricity prices. Through theory and examples we explain why, in the competitive equilibrium, sample-paths of prices can range from negative values, to values far beyond the "choke-up" price – which is usually considered to be the maximum price consumers are willing to pay. We also find that the variance of prices may be very large, but this variance decreases with increasing demand response.

1.1 Introduction

Electricity markets are intended to fulfill a range of goals: the pricing of electricity, the prescription of supply and demand-side decisions, and the creation of incentives for enhanced services and better technologies for electricity generation. The earliest efforts at deregulation appear to have been in Latin America in the 1980s, and subsequently in the UK in 1990. In the late 1990s, electricity markets were created in various U.S. zones, such as in California, New York, New England and the Pennsylvania-Jersey-Maryland (PJM) interchange [33]. The last decade has witnessed many changes in the

---

G. Wang, M. Negrete-Pincetic, A. Kowli, E. Shafieepoorfard and S. Meyn are with the CSL and the ECE department while U. V. Shanbhag is with the ISE department, at the University of Illinois at Urbana-Champaign, IL 61820 USA. e-mail: meyn@illinois.edu

design and organization of electricity markets both within the United States and beyond.

An electricity market may be viewed as a coupling between two constrained and highly complex dynamical systems, of which the first is purely physical while the second is financial.

- The physical system is a complex network comprising of power flowing through transmission lines, modulated by distributed generation units, Kirchhoff's laws, and operational and security constraints. Loads and generation are each subject to uncertainty. The stability of this network relies on instantaneous balancing between the production and consumption of electricity. Achieving this balance in a constrained environment is challenging due to the combination of uncertainty and a wide range of system constraints.
- The financial system is typically comprised of a coupled set of markets, such as the forward, day-ahead and real-time auctions. A sequence of market clearings leads to an associated trajectory of electricity prices for a specific type of auction. The financial system is also subject to both constraints and uncertainty: Constraints from the market structure itself, and uncertainty in the form of unpredictable fuel prices, volatility in demand, and supply-side stochasticity (e.g., via wind-based resources). The dynamics of the financial system can be dramatic: prices may vary by two orders of magnitude within short durations — A few examples are shown in Fig. 1.1.

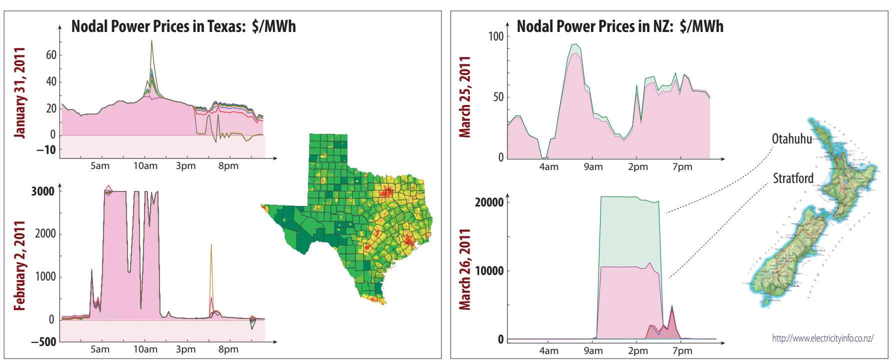

Fig. 1.1 Electricity prices in Texas and New Zealand in 2011

The presence of such complicating factors in these coupled systems, along with the often orthogonal relationship between societal needs and the economic goals of the market players, make electricity market design a challenging proposition.

Given the importance of an efficient and reliable grid infrastructure, the modeling and subsequent analysis of electricity markets has been a dominant area of research. Static equilibrium analysis via competitive equilibrium models [14], supply-function models [2, 4, 19] and Nash-Cournot models [13, 23] have provided insights regarding the nature of equilibria. Of these, supply-function equilibrium problems often lead to infinite dimensional variational problems, but conjectured [10] and parameterized [17] variants lead to more tractable problems. Extensions to more representative regimes that incorporate forward and day-ahead markets were examined in [16, 28, 35]. Dynamic generalizations have seen far less study in strategic settings, with recent work on dynamic Nash-Cournot models by Mookherjee et al. [25] being an exception. In general, the implication of dynamics in market analysis has received limited attention, in spite of the remarkable volatility seen in electricity markets around the world. This chapter surveys modest beginnings in the area of dynamic equilibrium analysis, drawing mainly from [6, 24, 30], following Cho & Meyn [7, 27], and also inspired by recent research reported in [18, 31, 36].

In the U.S., price swings observed in California in 2000-2001 are the most famous examples of price volatility. A few years prior, unexpected and equally dramatic price patterns brought down the Illinois electricity market [12]. More recent examples are illustrated in Fig. 1.1. Shown on the left are prices in ERCOT (the Texas market) for two days in 2011. January 31 was a typical day, with prices ranging from a high near \$80/MWh, and a low that was just below zero. Two days later on February 2, 2011, unusually cold weather in Texas resulted in real-time prices hitting the price cap of \$3,000, which is about 100 times the average price of \$30/MWh. Shown on the right is a far more dramatic example drawn from New Zealand in March, 2011, where electricity prices exceeded \$20,000/MWh in one region of the country, and remained near this extraordinary level for about six hours. These two incidents resulted in investigations in their respective communities [15, 32]. In New Zealand, the Electricity Authority has recently responded by retroactively reversing the prices to roughly one-tenth of the peak value, since these high prices threatened to "undermine confidence in, and ... damage the integrity and reputation of the wholesale electricity market."

While it is possible that strategic behavior and manipulation resulted in the wild price swings observed in Texas and New Zealand, in this chapter we demonstrate that such price patterns are *not* inconsistent with the *efficient* competitive equilibrium, where such manipulation is ruled out by assumption.

In a broad survey of reliability and market issues in the grid [9], Joe Chow and his coauthors assert that "*System reliability is an integral part of a properly designed deregulated electricity market, even though wholesale energy prices are its most visible piece*". Models that capture dynamics, constraints, and volatility are needed to gain any understanding of how to create markets that enhance reliability, with stable price profiles, in the face of uncertain generation assets and volatile demand. Understanding the coupled dynamics of the physical grid overlaid by a set of sequentially occurring markets is especially important today: There are incentives from the government to make wider use of "smart meters" to create a "smart grid". Concomitantly, there has been an impetus to install greater levels of intermittent renewable energy from sun, wind, and waves, which will bring greater uncertainty to the grid. Until some intelligence can be introduced to accommodate the increased complexity and uncertainty that will come with these changes, we believe that the term *entropic grid* best describes the power grid envisioned by policy makers and researchers today in 2011 [26]. A basic message to the power system community is that while we recognize the potential benefits of the Smart Grid vision, we must be aware of the potential issues arising from its entropic characteristics. With proper design, this uncertainty can be reduced, and the term *Smart Grid* will be justified.

The focus of this chapter is on developing models to capture the dynamics associated with the market side of the entropic grid. Our goal is to contribute to the understanding of the impact of dynamics, constraints, and uncertainty on dynamic competitive equilibria; factors which we expect to become more acute with the increased deployment of renewable resources. Such sentiment is aptly conveyed by Smith et. al. in the recent article [29], where the authors write that, "*little consideration was given to market design and operation under conditions of high penetrations of remote, variable renewable generation, such as wind … and solar energy, which had not yet appeared on the scene in any significant amounts.*" Our approach is the development of models that are able to characterize the competitive equilibria for a power network model that captures these complexities. This may be regarded as a stepping stone towards the creation of reliable markets for a smart grid.

We present a competitive economic equilibrium model that refines standard economic models (e.g., [1]), by including dynamics, uncertainty in supply and demand, and operational constraints associated with generation and transmission. Using a Lagrangian decomposition that is standard in static economic analysis and certain dynamic economic analyses [8, 22], we provide conditions for the existence of a competitive equilibrium in this dynamic setting, and its optimality with regard to suitably defined social planner's problem. Many of the conclusions obtained may be predicted from classical economic theory. In particular, under general conditions, the average equilibrium price coincides with the average marginal cost. However, in a competitive equilibrium, the sample path behavior can be as volatile as seen in the examples shown in Fig. 1.1. Moreover, in the presence of transmission constraints, equilibrium prices may become negative or they may exceed beyond the "choke-up" price that was predicted in [7].

The chapter also contains a brief economic analysis of a market with mixed generation sources, in which fast responding, expensive ancillary services are available to improve reliability. We consider a single example based on [6], where the solution to the "social planner's problem" was obtained, but a market analysis was not considered. We illustrate through numerical experiments that volatile price patterns result in very large variance, which can negatively impact ancillary service providers. We explore demand response as a potential solution and demonstrate its effectiveness in reducing price variance. We contend that models and concepts surveyed in this chapter will open new pathways for accommodating uncertainty, dynamics, and strategic behavior in electricity market models.

The remainder of this chapter contains three additional sections and is organized as follows. In Sec. 1.2, we present the economic and physical models for the electricity market players in a dynamic setting, and illustrate the models through representative test case examples. We devote Sec. 1.3 to the characterization of the competitive equilibrium of the dynamic electricity market model using a control-theoretic scaffolding. In particular, we derive conditions for the existence of a competitive equilibrium in terms of duality concepts from optimization theory. We also prove the two celebrated theorems of welfare economics, thus establishing the efficiency of market equilibria. In Sec. 1.4, we characterize the equilibrium prices for the dynamic market, and show that average prices coincide with average marginal cost under some general assumptions. However, the test case examples emphasize the volatility observed in the sample paths of prices as well as the price range, which reaches both negative and extremely large positive values. We provide concluding remarks and final thoughts in Sec. 1.5.

1.2 Electricity Market Model

Electricity markets are driven by economic objectives of the market participants as well as the reliability constraints associated with the physical limitations on generation and transmission. Hence, we explicitly consider both the economic objectives as well as the reliability constraints in our modeling.

The market model described here is a high-level abstraction of the electric industry. It is an extension of the equilibrium models found in standard economic texts, suitably modified to accommodate dynamics, physical constraints on generation and transmission, and uncertainty in supply and demand. The model consists of three "players": As in typical economic analyses, the two main players are the consumers and the suppliers representing the utility companies and generation owners, respectively. For simplicity, we restrict the discussion to a single consumer and a single supplier that respectively represent aggregation of all utility companies and generators across the grid. The third player is the *network*. The network is introduced as a player to capture the impacts of transmission constraints and exploit the network structure. We can think of this third player as corresponding to the independent system operator[1] (ISO) that operates the transmission grid in most electricity markets in the world. In what follows, we discuss the modeling of the three players and motivate the modeling aspects through examples.

---

[1] The ISO is an entity independent of the consumers and the suppliers that coordinates, controls and monitors a large electric power transmission grid and its associated electricity markets.

1.2.1 The Players

The power grid is represented by a graph in which each node represents a bus (which corresponds to a specific area/location), and each link represents a transmission line. There are N nodes, indexed as $\{1, \dots, N\}$, and L transmission lines, indexed as $\{1, \dots, L\}$. The network is assumed to be connected. A lossless DC model is used to characterize the relationship between nodal generation and demand, and power flow across the various links. For simplicity, throughout most of this chapter we assume that at each node there is exactly one source of generation, and one exogenous demand.

The consumer: We denote by $D_n(t)$ the demand at bus n at time t, and by $E_{\mathsf{D}n}(t)$ the energy withdrawn by the consumer at that bus. We assume that there is no free disposal for energy, which requires that $E_{\mathsf{D}n}(t) \le D_n(t)$ for all t. At time t, if sufficient generation is available at bus n, then $E_{\mathsf{D}n}(t) = D_n(t)$. In the event of insufficient generation, we have $E_{\mathsf{D}n}(t) < D_n(t)$, i.e., the consumer's demand is not met and he experiences a forced blackout.

The consumer must pay for energy consumed: The price[2] at bus n is denoted by $P_n(t)$. We use $D(t)$, $E_{\mathsf{D}}(t)$, and $P(t)$ to denote the associated N-dimensional column vectors, and we use bold face font to denote the entire sample path. For instance, $\boldsymbol{P} := \{P(t) : t \ge 0\}$.

The consumer obtains utility for energy consumption and disutility when demand is not met: This is modeled by the following two functions,

$$\text{Utility of consumption:} \qquad v_n(E_{\mathsf{D}n}(t)) \,, \tag{1.1a}$$

$$\text{Disutility of blackout:} \quad c_n^{\mathsf{bo}}(D_n(t) - E_{\mathsf{D}n}(t)) \,. \tag{1.1b}$$

The welfare of the consumer at time t is the signed sum of benefits and costs:

$$\mathcal{W}_{\mathsf{D}}(t) := \sum_n \bigl[v_n(E_{\mathsf{D}n}(t)) \;-\; c_n^{\mathsf{bo}}(D_n(t) - E_{\mathsf{D}n}(t)) \;-\; P_n(t)E_{\mathsf{D}n}(t)\bigr] \,. \tag{1.2}$$

The supplier: We denote by $E_{\mathsf{S}n}(t)$ and $R_{\mathsf{S}n}(t)$ the energy and reserve produced by the supplier at bus n at time t. The generation capacity $\boldsymbol{G}_{\mathsf{S}}$ available online coincides with $\boldsymbol{E}_{\mathsf{S}} + \boldsymbol{R}_{\mathsf{S}}$. The operational and physical constraints on available generation are expressed abstractly as

$$(\boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}}) \in \mathbf{X}_{\mathsf{S}} \,. \tag{1.3}$$

These constraints include minimum up/down-time constraints, ramping constraints and capacity constraints imposed by the physical limitations on generation.

At each time t, the supplier incurs costs for producing energy as well as maintaining reserves, which are represented as,

---

[2] Observe that prices may vary by the location. In the language of electricity markets, they are *locational prices*.

$$\text{Cost of energy:}\ \ c_n^{\mathsf{E}}(E_{\mathsf{S}n}(t))\,, \tag{1.4a}$$

$$\text{Cost of reserve:}\ \ c_n^{\mathsf{R}}(R_{\mathsf{S}n}(t))\,. \tag{1.4b}$$

The supplier at bus n receives the revenue $P_n E_{\mathsf{S}n}(t)$ for producing energy. The welfare of the supplier at time t is the difference between the supplier's revenue and costs,

$$\mathcal{W}_{\mathsf{S}}(t) := \sum_n \big[P_n E_{\mathsf{S}n}(t) - c_n^{\mathsf{E}}(E_{\mathsf{S}n}(t)) - c_n^{\mathsf{R}}(R_{\mathsf{S}n}(t)) \big]\,. \tag{1.5}$$

The network: The network player can be thought of as a *broker*, who buys energy from the supplier and sells it to the consumer. The transactions brokered by the network player are subject to the physical constraints of the transmission grid. The first constraint is based on the assumption that the network is lossless, so it neither generates nor consumes energy. Consequently, the network transactions are subject to the supply-demand balance constraint,

$$1^T E_{\mathsf{S}}(t) = 1^T E_{\mathsf{D}}(t) \qquad \text{for } t \geq 0\,. \tag{1.6}$$

The next set of constraints are due to the limitations of power flow through transmission lines. We adopt the DC power flow model [34] for the power flows through the lines. Without loss of generality, bus 1 is selected as the reference bus and the *injection shift factor matrix* $H \in [-1,1]^{N\times L}$ is defined. Note that H_{nl} represents the power that is distributed on line l when 1 MW is injected into bus n and withdrawn at the reference bus. If f_l^{max} denotes the capacity of transmission line l and $H_l \in \mathbb{R}^N$ represents the l-th column of H, the power flow constraint for line l can be expressed as,

$$-f_l^{\mathsf{max}} \leq [E_{\mathsf{S}}(t) - E_{\mathsf{D}}(t)]^T H_l \leq f_l^{\mathsf{max}} \qquad \text{for } t \geq 0\ . \tag{1.7}$$

We find it convenient to introduce a "network welfare function" to define a competitive equilibrium for the dynamic market model. The welfare of the network at time t represents the "brokerage charges" and is defined by,

$$\mathcal{W}_{\mathsf{T}}(t) := \sum_n \big[P_n (E_{\mathsf{D}n}(t) - E_{\mathsf{S}n}(t)) \big]\,. \tag{1.8}$$

The introduction of the network welfare function is purely for the sake of analysis. In Sec. 1.3 we assume that the "network" maximizes its welfare, subject to the supply-demand balance constraint (1.6) and power flow constraint (1.7) for each line l, which are collectively summarized by the notation,

$$(\boldsymbol{E}_{\mathsf{D}}, \boldsymbol{E}_{\mathsf{S}}) \in \mathbf{X}_{\mathsf{T}}\,. \tag{1.9}$$

1.2.2 Test Case Examples

The following three examples are intended to illustrate the modeling conventions described in the preceding section. We return to these in Sec. 1.4 to illustrate the conclusions of Sec. 1.3.

Example A – Single-Bus Model: In this simple model there is a single generator, a single consumer, and no transmission network. The exogenous demand $\boldsymbol{D}$ is scalar valued, and the supply-demand balance constraints amount to $\boldsymbol{E} := \boldsymbol{E}_{\mathsf{S}} = \boldsymbol{E}_{\mathsf{D}}$. This example is similar to the model of [7].

We impose the simplest constraints on generation — those imposed by limitations on ramping capabilities. The positive constants ζ^+ and ζ^- represent the maximum rates for ramping up and down the generators: For each $t_1 > t_0 \geq 0$,

$$-\zeta^- \leq \frac{E_{\mathsf{S}}(t_1) - E_{\mathsf{S}}(t_0)}{t_1 - t_0} + \frac{R_{\mathsf{S}}(t_1) - R_{\mathsf{S}}(t_0)}{t_1 - t_0} \leq \zeta^+ . \tag{1.10}$$

The constraint set $\mathbf{X}_{\mathsf{S}}$ consists of all pairs $(\boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}})$ which satisfy (1.10).

Piecewise-linear utility, disutility and cost functions take the following form,

$$\begin{aligned} \text{Utility of consumption: } & vE(t)\,, \\ \text{Disutility of blackout: } & c^{\mathsf{bo}} \max(D(t) - E(t), 0)\,, \\ \text{Cost of generation: } & cE(t) + cR(t)\,; \end{aligned} \tag{1.11}$$

where v, c^{bo} and c are positive constants. The parameter c^{bo} is the cost of outage incurred by the consumers for each unit of demand that is not satisfied. This may be tens of thousands of dollars per megawatt hour, depending on the location and time of day. In typical static analysis, the quantity $(v + c^{\mathsf{bo}})$ represents the maximum price the consumer is willing to pay for electricity, and is known as the *choke-up* price [7].

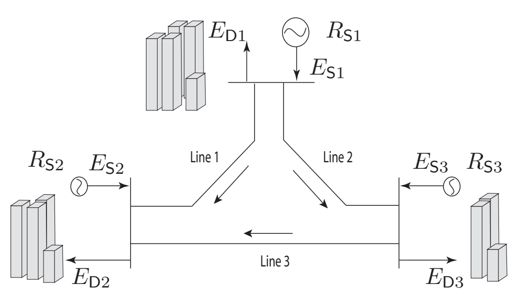

Fig. 1.2 Texas model: A 3-bus network.

Example B – Texas Model: The network topology is shown in Fig. 1.2. There are sources of demand and supply at each of the three nodes shown in the figure, and each node is connected to the other nodes via transmission lines.

The supply-demand balance constraints (1.6) result in the single equation,

$$E_{\mathsf{S1}}(t) + E_{\mathsf{S2}}(t) + E_{\mathsf{S3}}(t) = E_{\mathsf{D1}}(t) + E_{\mathsf{D2}}(t) + E_{\mathsf{D3}}(t) \quad \text{for } t \geq 0\,. \tag{1.12}$$

Suppose that the impedances of all three transmission lines are identical. Further, the arrows indicate the direction of power flow which is assumed to be positive. Then, with bus 1 chosen as the reference bus, the injection shift factor matrix is given by

$$H = \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ -2 & -1 & -1 \\ -1 & -2 & 1 \end{bmatrix}. \tag{1.13}$$

The power flow constraints for the three transmission lines are expressed as

$$-F^{\mathsf{max}} \;\leq\; \begin{bmatrix} E_{\mathsf{S1}}(t) - E_{\mathsf{D1}}(t) \\ E_{\mathsf{S2}}(t) - E_{\mathsf{D2}}(t) \\ E_{\mathsf{S3}}(t) - E_{\mathsf{D3}}(t) \end{bmatrix}^T H \;\leq\; F^{\mathsf{max}} \qquad \text{for } t \geq 0\,, \tag{1.14}$$

where $F^{\mathsf{max}} = [f_{12}^{\mathsf{max}}, f_{13}^{\mathsf{max}}, f_{23}^{\mathsf{max}}]$ with f_{mn}^{max} representing the capacity limit of each line $l = 1, 2, 3$ connected between nodes m and n. Then, the network constraints set $\mathbf{X}_{\mathsf{T}}$ consists of all sample paths $(\boldsymbol{E}_{\mathsf{D}}, \boldsymbol{E}_{\mathsf{S}})$ which satisfy (1.12) and (1.14).

Example C – Primary and Ancillary Services: Although we have restricted the analysis to a single source of generation at each node, the physical characteristics of generation may vary by node and also for generators located at one node. In the case of two generation sources, and with network constraints relaxed, we arrive at a model similar to the dynamic newsboy model for generation introduced in [6].

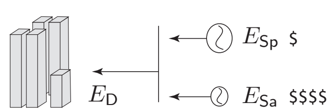

Fig. 1.3 Primary and Ancillary Service

Example C is used to illustrate the role of *ancillary service* in power systems operation. In power markets operating in the world today, the forecast demand is often met by a primary service – the cheapest source of service capacity – through a prior contract. Nuclear and coal generators are typical primary service providers. The deviations from forecast demand can be met through primary service, as well as ancillary service. Ancillary service may be costly, but it can be ramped up/down at a much faster rate than the primary service. Gas turbine generators are an example of the more expensive, yet more responsive sources of ancillary service. The simplified model for a system with primary and ancillary generators is depicted in Fig. 1.3.

We denote by $G_{\mathsf{p}}(t)$ and $G_{\mathsf{a}}(t)$ the available online capacity of primary and ancillary generators at time t. Note that $G_{\mathsf{p}}(t)$ represents the deviations in primary service from the day-ahead schedules and, hence, can be less than zero while $G_{\mathsf{a}}(t) \geq 0$. The demand at time t is given as $D(t)$ and the reserve at that time is,

$$R(t) = G_{\mathsf{p}}(t) + G_{\mathsf{a}}(t) - D(t), \quad t \geq 0\,. \tag{1.15}$$

The two sources of generation are distinguished by their *ramping* capabilities: ζ_{p}^{+}, ζ_{a}^{+}, ζ_{p}^{-} and ζ_{a}^{-} represent the maximum rates for ramping up and down the primary and ancillary services, respectively. The ramping limits imposed on $G_{\mathsf{p}}(t)$ and $G_{\mathsf{a}}(t)$ take the form similar to (1.10). We assume that $\zeta_{\mathsf{a}}^{+} > \zeta_{\mathsf{p}}^{+}$ and $\zeta_{\mathsf{a}}^{-} > \zeta_{\mathsf{p}}^{-}$ to reflect the ability of ancillary service to ramp up faster than primary service. Similarly, with c^{p} and c^{a} representing the per unit production costs of primary and ancillary services, respectively, we assume $c^{\mathsf{a}} \gg c^{\mathsf{p}}$ to emphasize that ancillary service is more expensive than primary service.

1.3 Competitive Equilibria and Efficiency

The competitive equilibrium is used in economic analysis as a vehicle to study the outcomes of a market under a set of idealized assumptions [1, 3, 22]. It is characterized by the allocation of goods and their prices: for the given prices, each player maximizes its welfare subject to constraints on production/consumption of the goods. That is, the competitive equilibrium characterizes an allocation that is optimal from the point of view of individual players. In this theory, the solution to the so-called *social planner's problem* (SPP) serves as a benchmark and is an allocation of goods that maximizes the economic well-being of the aggregation of all players, and is optimal from the point of view of the entire system. This optimal solution is known as an *efficient allocation*.

The fundamental theorems of welfare economics provide a strong link between competitive equilibria and the SPP. The first theorem states that any competitive equilibrium leads to an efficient allocation, while the second states the converse. In this section, we revisit the concepts of competitive equilibrium and efficiency for the dynamic electricity market model presented in Sec. 1.2, and establish the fundamental welfare theorems for such a market.

1.3.1 Preliminaries

The usual definition of a competitive equilibrium with two players is based on the respective optimization problems of the consumer and the supplier. We adopt the same convention in the dynamic setting, but we extend the equilibrium definition to accommodate the third player – the network. In the dynamic setting, we adopt the long-run discounted expected surplus as the consumer's objective function. With discount rate γ, the long-run discounted expected consumer welfare is

$$K_{\mathsf{D}} := \mathsf{E}\Big[\int e^{-\gamma t}\mathcal{W}_{\mathsf{D}}(t)\,dt\Big]\,.$$

The long-run discounted welfare of the supplier and the network are defined similarly, denoted K_{S}, K_{T}, respectively.

The consumer, supplier, and network each aim to optimize their respective mean discounted welfare K_{D}, K_{S}, and K_{T}. In general, these quantities depend on the initial condition of the system; we suppress this dependency whenever possible.

The assumptions imposed in this dynamic setting are intended to mirror those used in equilibrium theory for static economic models. To emphasize the similarities, we adopt the following Hilbert-space notation: For two real-valued stochastic processes $\boldsymbol{F}$ and $\boldsymbol{G}$, we introduce the inner product,

$$\langle \boldsymbol{F}, \boldsymbol{G}\rangle := \mathsf{E}\Big[\int e^{-\gamma t}F(t)G(t)\,dt\Big]\,, \tag{1.16}$$

and write $\boldsymbol{F} \in L_2$ if $\langle \boldsymbol{F}, \boldsymbol{F}\rangle < \infty$. For example, using this notation, we have $K_{\mathsf{D}} = \langle \boldsymbol{\mathcal{W}}_{\mathsf{D}}, \mathbf{1}\rangle$, where $\mathbf{1}$ denotes the process that is identically unity. It is assumed throughout that the components of the vector-valued processes $\{\boldsymbol{E}_{\mathsf{D}}, \boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}}, \boldsymbol{P}\}$ belong to L_2.

In the formal definition of a competitive equilibrium given in Def. 1 below, the optimization problems of the supplier and of the network are each subject to physical constraints. However, as in [7], the set of feasible strategies for the consumer are *not* subject to the constraints on generation or transmission specified in (1.3, 1.9).

We impose several additional assumptions for the dynamic market model, each of which is an extension of what is typically assumed in the competitive equilibrium analysis of a static model.

(A1) Consumers and suppliers share equal information. This is modeled using a common *filtration*: an increasing family of σ-algebras, denoted

$\mathcal{H} = \{\mathcal{H}_t : t \geq 0\}$. The demand process $\boldsymbol{D}$, the price process $\boldsymbol{P}$, and the decisions of the consumer $\boldsymbol{E}_{\mathsf{D}}$ and the supplier $(\boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}})$ are adapted to this filtration.

(A2) The components of these vector-valued processes lie in L_2: Consumer decisions $\boldsymbol{E}_{\mathsf{D}}$, supplier decisions $(\boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}})$, and the price process $\boldsymbol{P}$. Moreover, utility and cost functions are subject to quadratic growth: These functions are non-negative, and for each n, there exists $k_n < \infty$ such that for each scalar r, and each $e \geq 0$,

$$
\begin{aligned}
v_n(e) &\leq k_n(1+e^2) \qquad & c_n^{\mathsf{E}}(e) &\leq k_n(1+e^2) \\
c_n^{\mathsf{bo}}(r) &\leq k_n(1+r^2) \qquad & c_n^{\mathsf{R}}(r) &\leq k_n(1+r^2)
\end{aligned}
$$

(A3) Prices are *exogenous* in the following sense: for each $t_0 > 0$, the future prices $\{P(t) : t > t_0\}$ are conditionally independent of $\{E_{\mathsf{D}}(t), E_{\mathsf{S}}(t) : t \leq t_0\}$, given current and past prices $\{P(t) : t \leq t_0\}$.

Under assumption (A3), also known as the *price-taking assumption*, no market player is large enough to influence prices and, hence, market manipulation is eliminated.

A competitive equilibrium for the dynamic market model is defined as follows.

Definition 1. A *competitive equilibrium* is a quadruple of process vectors: consumed energy, supplied energy, supplied reserve, and energy price; denoted as $\{\boldsymbol{E}^e_{\mathsf{D}}, \boldsymbol{E}^e_{\mathsf{S}}, \boldsymbol{R}^e_{\mathsf{S}}, \boldsymbol{P}^e\}$, which satisfies the following conditions:

$\boldsymbol{E}^e_{\mathsf{D}}$ solves the consumer's optimization problem;

$$
\boldsymbol{E}^e_{\mathsf{D}} \in \arg\max_{\boldsymbol{E}_{\mathsf{D}}} \langle \boldsymbol{\mathcal{W}}_{\mathsf{D}}, \mathbf{1} \rangle \,. \tag{1.17}
$$

$(\boldsymbol{E}^e_{\mathsf{S}}, \boldsymbol{R}^e_{\mathsf{S}})$ solves the supplier's optimization problem;

$$
\begin{aligned}
(\boldsymbol{E}^e_{\mathsf{S}}, \boldsymbol{R}^e_{\mathsf{S}}) &\in \arg\max_{\boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}}} \langle \boldsymbol{\mathcal{W}}_{\mathsf{S}}, \mathbf{1} \rangle \\
\text{subject to} &\quad \textit{generation constraints } (1.3)
\end{aligned} \tag{1.18}
$$

$(\boldsymbol{E}^e_{\mathsf{D}}, \boldsymbol{E}^e_{\mathsf{S}})$ solves the network's optimization problem with $\boldsymbol{P}^e$ being the Lagrange multiplier associated with (1.9),

$$
\begin{aligned}
(\boldsymbol{E}^e_{\mathsf{D}}, \boldsymbol{E}^e_{\mathsf{S}}) &\in \arg\max_{\boldsymbol{E}_{\mathsf{D}}, \boldsymbol{E}_{\mathsf{S}}} \langle \boldsymbol{\mathcal{W}}_{\mathsf{T}}, \mathbf{1} \rangle \\
\text{subject to} &\quad \textit{network constraints } (1.9)
\end{aligned} \tag{1.19}
$$

The consumer, supplier and network are also subject to the measurability constraint outlined in assumption (A1) in their respective optimization problems. □

As discussed at the start of this section, to evaluate the market, we introduce a *social planner* who aims to maximize the economic well-being of aggregate representation of the players in the system. We stress that there is no actual planner — this is another analytical device. The social planner uses the total welfare, denoted by $\mathcal{W}_{\mathsf{tot}}(t)$, to measure the economic well-being of the system with

$$\mathcal{W}_{\mathsf{tot}}(t) := \mathcal{W}_{\mathsf{S}}(t) + \mathcal{W}_{\mathsf{D}}(t) + \mathcal{W}_{\mathsf{T}}(t)\,. \tag{1.20}$$

The total welfare can be expressed,

$$\mathcal{W}_{\mathsf{tot}}(t) = \sum_n \left[v_n(E_{\mathsf{D}n}(t)) - c_n^{\mathsf{bo}}(D_n(t) - E_{\mathsf{D}n}(t)) - c_n^{\mathsf{E}}(E_{\mathsf{S}n}(t)) - c_n^{\mathsf{R}}(R_{\mathsf{S}n}(t)) \right]\,,$$

which is independent of the price process.

The efficient allocation is derived by solving the social planner's problem (SPP) described at the start of this section. It is formally defined as follows:

Definition 2. The social planner's problem is given by,

$$\begin{aligned} \max_{\boldsymbol{E}_{\mathsf{D}},\boldsymbol{E}_{\mathsf{S}},\boldsymbol{R}_{\mathsf{S}}} \quad & \langle \boldsymbol{\mathcal{W}}_{\mathsf{tot}}, \mathbf{1} \rangle \\ \text{subject to} \quad & \textit{generation constraints}\ (1.3)\,, \\ \text{and} \quad & \textit{network constraints}\ (1.9)\,. \end{aligned} \tag{1.21}$$

Its solution is called an *efficient allocation.* □

We assume that the SPP (1.21) has a solution, denoted by $(\boldsymbol{E}_{\mathsf{D}}^*, \boldsymbol{E}_{\mathsf{S}}^*, \boldsymbol{R}_{\mathsf{S}}^*)$.

Given these preliminaries, we have set the stage for the two fundamental theorems of welfare economics which link the above two definitions.

Theorem 1 (First Welfare Theorem). *Any competitive equilibrium, if it exists, is efficient.* □

Theorem 2 (Second Welfare Theorem). *If a market admits a competitive equilibrium, then any efficient allocation can be sustained by a competitive equilibrium.* □

The proofs of these results are contained in Sec. 1.3.2 that follows.

1.3.2 Analysis

The main result here is Theorem 3, that characterizes the existence of a competitive equilibrium. At the end of this section we establish the first and second fundamental theorems as corollaries to this result.

The analysis in this section is based on the Lagrangian relaxation framework presented in [30]. Lagrange multipliers are scalar processes, denoted

$(\boldsymbol{\lambda}, \boldsymbol{\mu}_l^+, \boldsymbol{\mu}_l^-)$, with $\boldsymbol{\lambda}$ unconstrained, and with $\mu_l^+(t) \geq 0$ and $\mu_l^-(t) \geq 0$ for all t and l. These processes are assumed to lie in L_2, and adapted to $\mathcal{H}$ (see (A1)). The Lagrangian of the SPP is denoted,

$$\mathcal{L} = -\langle \boldsymbol{\mathcal{W}}_{\mathsf{tot}}, \mathbf{1} \rangle + \langle \boldsymbol{\lambda}, (1^T \boldsymbol{E}_{\mathsf{D}} - 1^T \boldsymbol{E}_{\mathsf{S}}) \rangle + \sum_l \langle \boldsymbol{\mu}_l^+, (\boldsymbol{E}_{\mathsf{S}} - \boldsymbol{E}_{\mathsf{D}})^T H_l - f_l^{\mathsf{max}} \rangle$$
$$+ \sum_l \langle \boldsymbol{\mu}_l^-, -(\boldsymbol{E}_{\mathsf{S}} - \boldsymbol{E}_{\mathsf{D}})^T H_l - f_l^{\mathsf{max}} \rangle \,, \tag{1.22}$$

A key step is to define the candidate price process $\boldsymbol{P}$ as

$$P_n(t) := \lambda(t) + \sum_l (\mu_l^-(t) - \mu_l^+(t)) H_{ln}, \quad t \geq 0, \ n \geq 1. \tag{1.23}$$

The Lagrangian is then expressed as the sum of three terms, each of which is a sum over the N nodes,

$$\mathcal{L} = -\sum_n \{ \langle v_n(\boldsymbol{E}_{\mathsf{D}n}) - c_n^{\mathsf{bo}}(\boldsymbol{D}_n - \boldsymbol{E}_{\mathsf{D}n}), \mathbf{1} \rangle - \langle \boldsymbol{P}_n, \boldsymbol{E}_{\mathsf{D}n} \rangle \}$$
$$- \sum_n \{ \langle \boldsymbol{P}_n, \boldsymbol{E}_{\mathsf{S}n} \rangle - \langle c_n^{\mathsf{E}}(\boldsymbol{E}_{\mathsf{S}n}) + c_n^{\mathsf{R}}(\boldsymbol{R}_{\mathsf{S}n}), \mathbf{1} \rangle \} \ - \ \sum_l \langle \boldsymbol{\mu}_l^+ + \boldsymbol{\mu}_l^-, f_l^{\mathsf{max}} \rangle \,.$$

The first two terms correspond to $-\boldsymbol{\mathcal{W}}_{\mathsf{D}}$ and $-\boldsymbol{\mathcal{W}}_{\mathsf{S}}$ respectively (the negative consumer and supplier welfare functions), defined using the price $\boldsymbol{P}$ given in (1.23).

The dual functional for the SPP is defined as the minimum,

$$h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-) = \min_{\boldsymbol{E}_{\mathsf{D}}, \boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}}} \mathcal{L} \,. \tag{1.24}$$

The following weak duality bound follows since the minimization in (1.24) amounts to a relaxation of the SPP (1.21).

Lemma 1 (Weak Duality). *For any allocation* $\{\boldsymbol{E}_{\mathsf{D}}, \boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}}\}$ *and Lagrangian multiplier* $(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-)$ *with* $\boldsymbol{\mu}^+, \boldsymbol{\mu}^- \geq 0$*, we have*

$$-\langle \boldsymbol{\mathcal{W}}_{\mathsf{tot}}, \mathbf{1} \rangle \geq h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-) \,. \tag{1.25}$$

We say that *strong duality* holds if we have equality in (1.25).

Theorem 3 characterizes the existence of a competitive equilibrium in terms of strong duality.

Theorem 3 (Existence of Competitive Equilibria). *The market admits a competitive equilibrium if and only if the SPP satisfies strong duality.*

Proof. We first prove the *sufficient condition*: strong duality implies existence of competitive equilibrium. Since strong duality holds, we have

$$-\langle \boldsymbol{\mathcal{W}}_{\mathsf{tot}}, \mathbf{1} \rangle = h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-) \,. \tag{1.26}$$

Suppose that the allocation $\{\boldsymbol{E}_{\mathsf{D}}, \boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}}\}$ is feasible for the SPP. We then construct a competitive equilibrium with price as given in (1.23).

The feasibility of the triple $\{\boldsymbol{E}_{\mathsf{D}}, \boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}}\}$ for SPP implies $1^T E_{\mathsf{S}}(t) = 1^T E_{\mathsf{D}}(t)$ for all t, and hence

$$\mathcal{L} = -\langle \boldsymbol{\mathcal{W}}_{\mathsf{tot}}, \mathbf{1}\rangle + \sum_l \langle \boldsymbol{\mu}_l^+, (\boldsymbol{E}_{\mathsf{S}} - \boldsymbol{E}_{\mathsf{D}})^T H_l - f_l^{\mathsf{max}}\rangle + \sum_l \langle \boldsymbol{\mu}_l^-, -(\boldsymbol{E}_{\mathsf{S}} - \boldsymbol{E}_{\mathsf{D}})^T H_l - f_l^{\mathsf{max}}\rangle \,.$$

Furthermore, feasibility also implies $-f_l^{\mathsf{max}} \le (E_{\mathsf{S}}(t) - E_{\mathsf{D}}(t))^T H_l \le f_l^{\mathsf{max}}$, and given the non-negativity of $\boldsymbol{\mu}^+, \boldsymbol{\mu}^-$, we have

$$\langle \boldsymbol{\mu}^+, (\boldsymbol{E}_{\mathsf{S}} - \boldsymbol{E}_{\mathsf{D}})^T H_l - f_l^{\mathsf{max}}\rangle \le 0\,, \text{ and } \langle \boldsymbol{\mu}^-, -(\boldsymbol{E}_{\mathsf{S}} - \boldsymbol{E}_{\mathsf{D}})^T H_l - f_l^{\mathsf{max}}\rangle \le 0\,.$$

Therefore, using (1.26), we have $\mathcal{L} \le h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-)$. But, by the definition (1.24), we have $\mathcal{L} \ge h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-)$, so that we obtain the identity,

$$h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-) = \mathcal{L}\,. \tag{1.27}$$

This identity implies that $\boldsymbol{E}_{\mathsf{D}}$ maximizes the consumer's welfare, $\{\boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}}\}$ maximizes the supplier's welfare, and

$$\langle \boldsymbol{\mu}^+, (\boldsymbol{E}_{\mathsf{S}} - \boldsymbol{E}_{\mathsf{D}})^T H_l - f_l^{\mathsf{max}}\rangle = 0\,, \text{ and } \langle \boldsymbol{\mu}^-, -(\boldsymbol{E}_{\mathsf{S}} - \boldsymbol{E}_{\mathsf{D}})^T H_l - f_l^{\mathsf{max}}\rangle = 0\,.$$

Using the prices defined in (1.23), we compute the network's optimization objective as follows,

$$\begin{aligned}\Big\langle \sum_n [\boldsymbol{P}_n(\boldsymbol{E}_{\mathsf{D}n} - \boldsymbol{E}_{\mathsf{S}n})], \mathbf{1}\Big\rangle &= \Big\langle \sum_n \Big[\sum_l (\boldsymbol{\mu}_l^- - \boldsymbol{\mu}_l^+) H_{ln} (\boldsymbol{E}_{\mathsf{D}n} - \boldsymbol{E}_{\mathsf{S}n})\Big], \mathbf{1}\Big\rangle \\ &= \Big\langle \sum_l (\boldsymbol{\mu}_l^+ + \boldsymbol{\mu}_l^-) f_l^{\mathsf{max}}, \mathbf{1}\Big\rangle\,.\end{aligned}$$

Note that the network's objective function is independent of $\boldsymbol{E}_{\mathsf{D}}$ and $\{\boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}}\}$, which implies that the network welfare is maximized under the prices $\{\boldsymbol{P}_n\}$. Thus, we conclude that $\boldsymbol{P}$ as defined in (1.23) is the equilibrium price as claimed and that $\{\boldsymbol{E}_{\mathsf{D}}, \boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}}, \boldsymbol{P}\}$ constitute a competitive equilibrium.

Next we establish the *necessary condition*: existence of a competitive equilibrium implies strong duality. Suppose that $\{\boldsymbol{E}_{\mathsf{D}}^e, \boldsymbol{E}_{\mathsf{S}}^e, \boldsymbol{R}_{\mathsf{S}}^e, \boldsymbol{P}^e\}$ is a competitive equilibrium. Then we know that $\{\boldsymbol{E}_{\mathsf{D}}^e, \boldsymbol{E}_{\mathsf{S}}^e\}$ maximizes the network welfare when the price is $\boldsymbol{P}^e$. The Lagrangian associated with the network's optimization problem is expressed as follows: For any $\boldsymbol{\mu}^+, \boldsymbol{\mu}^- \ge 0$,

$$\begin{aligned}\mathcal{L}_{\mathsf{T}} = &- \sum_n \langle \boldsymbol{P}_n^e, (\boldsymbol{E}_{\mathsf{D}n}^e - \boldsymbol{E}_{\mathsf{S}n}^e)\rangle + \langle \boldsymbol{\lambda}, (1^T \boldsymbol{E}_{\mathsf{D}}^e - 1^T \boldsymbol{E}_{\mathsf{S}}^e)\rangle \\ &+ \sum_l \langle \boldsymbol{\mu}_l^+, (\boldsymbol{E}_{\mathsf{S}}^e - \boldsymbol{E}_{\mathsf{D}}^e)^T H_l - f_l^{\mathsf{max}}\rangle + \sum_l \langle \boldsymbol{\mu}_l^-, -(\boldsymbol{E}_{\mathsf{S}}^e - \boldsymbol{E}_{\mathsf{D}}^e)^T H_l - f_l^{\mathsf{max}}\rangle\,.\end{aligned} \tag{1.28}$$

The network's optimization problem is a linear program and, hence, the optimum $\{\boldsymbol{E}^e_{\mathsf{D}}, \boldsymbol{E}^e_{\mathsf{S}}\}$ satisfies the Karush-Kuhn-Tucker (KKT) conditions. As a consequence, associated with the constraints $\frac{\partial \mathcal{L}_{\mathsf{T}}}{\partial E_{\mathsf{D}n}} = \frac{\partial \mathcal{L}_{\mathsf{T}}}{\partial E_{\mathsf{S}n}} = 0$ at the optimum, there exist $\{\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-\}$ such that

$$\boldsymbol{P}^e_n = \boldsymbol{\lambda} + \sum_l (\boldsymbol{\mu}^-_l - \boldsymbol{\mu}^+_l) H_{ln}\,;$$

that is, equilibrium price process $\boldsymbol{P}^e_n$ satisfies (1.23). Moreover, by complementary-slackness, we have

$$\begin{aligned} \langle \boldsymbol{\lambda}, (1^T \boldsymbol{E}^e_{\mathsf{D}} - 1^T \boldsymbol{E}^e_{\mathsf{S}}) \rangle &= 0\,, \\ \langle \boldsymbol{\mu}^+, (\boldsymbol{E}^e_{\mathsf{S}} - \boldsymbol{E}^e_{\mathsf{D}})^T H_l - f^{\max}_l \rangle &= 0\,, \\ \langle \boldsymbol{\mu}^-, -(\boldsymbol{E}^e_{\mathsf{S}} - \boldsymbol{E}^e_{\mathsf{D}})^T H_l - f^{\max}_l \rangle &= 0\,. \end{aligned} \tag{1.29}$$

Suppose that the dual function $h(\cdot)$ for the SPP is formulated using the multipliers $\boldsymbol{\lambda}$, $\boldsymbol{\mu}^+$, and $\boldsymbol{\mu}^-$ from the network's optimization problem. Since $\{\boldsymbol{E}^e_{\mathsf{D}}, \boldsymbol{E}^e_{\mathsf{S}}, \boldsymbol{R}^e_{\mathsf{S}}, \boldsymbol{P}^e\}$ is a competitive equilibrium, $\boldsymbol{E}^e_{\mathsf{D}}$ maximizes the consumer's welfare, and $\{\boldsymbol{E}^e_{\mathsf{S}}, \boldsymbol{R}^e_{\mathsf{S}}\}$ maximizes the supplier's welfare. Based on the form (1.23) for price process $\boldsymbol{P}^e$, we conclude that

$$\{\boldsymbol{E}^e_{\mathsf{D}}, \boldsymbol{E}^e_{\mathsf{S}}, \boldsymbol{R}^e_{\mathsf{S}}\} \in \operatorname*{arg\,min}_{\boldsymbol{E}_{\mathsf{D}}, \boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}}} \mathcal{L}\,.$$

Substituting $\{\boldsymbol{E}^e_{\mathsf{D}}, \boldsymbol{E}^e_{\mathsf{S}}, \boldsymbol{R}^e_{\mathsf{S}}\}$ into (1.22), and applying the complementary slackness conditions from (1.29), we have

$$-\langle \boldsymbol{\mathcal{W}}_{\mathsf{tot}}, \mathbf{1} \rangle = h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-)\,.$$

That is, strong duality holds. ☐

We stress that Theorem 3 characterizes prices in any competitive equilibrium.

Corollary 1. *The only candidates for prices in a competitive equilibrium are given by* (1.23)*, based on the optimal Lagrange multipliers.*

Proof. This is because only the optimal Lagrange multipliers could possibly lead to strong duality. ☐

Thus, Theorem 3 tells us that computation of prices and quantities can be decoupled: The quantities $\{\boldsymbol{E}_{\mathsf{D}}, \boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}}\}$ are obtained through the solution of the SPP, and the price process $\boldsymbol{P}^e$ is obtained as a solution to its dual. The following corollary underlines the fact that if $\boldsymbol{P}^e$ supports one competitive equilibrium, then it supports any other competitive equilibrium.

Corollary 2. *If* $\{\boldsymbol{E}^1_{\mathsf{D}}, \boldsymbol{E}^1_{\mathsf{S}}, \boldsymbol{R}^1_{\mathsf{S}}, \boldsymbol{P}^1\}$ *and* $\{\boldsymbol{E}^2_{\mathsf{D}}, \boldsymbol{E}^2_{\mathsf{S}}, \boldsymbol{R}^2_{\mathsf{S}}, \boldsymbol{P}^2\}$ *are two competitive equilibria, then* $\{\boldsymbol{E}^2_{\mathsf{D}}, \boldsymbol{E}^2_{\mathsf{S}}, \boldsymbol{R}^2_{\mathsf{S}}, \boldsymbol{P}^1\}$ *is also a competitive equilibrium.*

Proof. Due to the necessary condition of Theorem 3, there exist Lagrange multipliers $(\boldsymbol{\lambda}_1, \boldsymbol{\mu}_1^+, \boldsymbol{\mu}_1^-)$ and $(\boldsymbol{\lambda}_2, \boldsymbol{\mu}_2^+, \boldsymbol{\mu}_2^-)$ corresponding to the two equilibria, such that

$$h(\boldsymbol{\lambda}_1, \boldsymbol{\mu}_1^+, \boldsymbol{\mu}_1^-) = -\langle \boldsymbol{\mathcal{W}}_{\mathsf{tot1}}, \mathbf{1} \rangle = -\langle \boldsymbol{\mathcal{W}}_{\mathsf{tot2}}, \mathbf{1} \rangle \,.$$

By the sufficient condition of Theorem 3, any of these price and quantity pairs satisfying strong duality will constitute a competitive equilibrium. □

Proof of Theorem 1. From the proof of necessary condition of Theorem 3, for any competitive equilibrium $\{\boldsymbol{E}_{\mathsf{D}}^e, \boldsymbol{E}_{\mathsf{S}}^e, \boldsymbol{R}_{\mathsf{S}}^e, \boldsymbol{P}^e\}$, there exist dual variables $(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-)$ such that

$$h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-) = -\langle \boldsymbol{\mathcal{W}}_{\mathsf{tot}}, \mathbf{1} \rangle \,.$$

From weak duality (1.25), we know that both the SPP and its dual problem are optimized. Therefore, $\{\boldsymbol{E}_{\mathsf{D}}^e, \boldsymbol{E}_{\mathsf{S}}^e, \boldsymbol{R}_{\mathsf{S}}^e\}$ is an efficient allocation. □

Proof of Theorem 2. We assume that a competitive equilibrium exists. Then, applying Theorem 1, there exists at least one equilibrium price process $\boldsymbol{P}^e$ that supports one of the efficient allocations. By Theorem 3, the existence of competitive equilibrium implies strong duality. Consequently, from the sufficient condition of Theorem 3, any of these price and quantity pairs satisfying strong duality will constitute a competitive equilibrium. In other words, the price $\boldsymbol{P}^e$ supports all efficient allocations. □

1.4 Equilibrium Prices for the Test Cases

The dynamic electricity market model of Sec. 1.2 is an appropriate representation of a competitive market as analyzed in a standard economics text. The dynamic nature of the constraints on generation may lead to volatile prices that negatively impact the consumers, the suppliers, or both. The sample path behavior of prices in a competitive equilibrium can look as erratic as the worst days during the crises in Illinois or California in the 1990s, or in Texas and New Zealand during the early months of 2011.

In this section, we characterize the equilibrium prices for the test case examples presented in Sec. 1.2.2 using the Lagrangian relaxation framework of Sec. 1.3. We find that even in the dynamic setting, market outcomes reflect the standard conclusions for efficient markets: Prices equal marginal costs, but only *on average*. We also illustrate how binding transmission constraints can result in prices that are negative or that exceed the choke-up price. Finally, we investigate the characteristics of prices for fast responding, expensive ancillary services introduced in Example C and show, through numerical experiments, that volatile price patterns result in very large variance of price.

1.4.1 Single-bus Model

Here we investigate sample path prices and their mean value for Example A — the single-bus model. Recall that the generation constraint set X_S is defined by the ramp constraints (1.10), and piecewise-linear forms are assumed for the utility and cost functions. Also, supply-demand balance necessitates that $\boldsymbol{E} := \boldsymbol{E}_\mathsf{S} = \boldsymbol{E}_\mathsf{D}$.

A special case of Example A is the model of [7], wherein the equilibrium price is obtained as

$$P^e(t) = (v + c^{\mathsf{bo}})\mathbb{I}\{R^*(t) < 0\}, \tag{1.30}$$

with $\boldsymbol{R}^*$ denoting the reserve process in the solution of the SPP. The quantity $(v + c^{\mathsf{bo}})$ is the choke-up price, which can be extremely large in a real power system. Consequently, equilibrium prices show tremendous volatility. However, when initial reserves are sufficiently large, the *average* price coincides with marginal cost c for generation, in the sense that

$$\gamma\mathsf{E}\Big[\int e^{-\gamma t} P^e(t)\, dt\Big] = c\,. \tag{1.31}$$

These conclusions were first obtained in [7] through direct calculation, based on specific statistical assumptions on demand. We show here that the same conclusions can be derived in far greater generality based on a Lagrangian relaxation technique.

We first establish the formula for $\boldsymbol{P}^e$.

Proposition 1. *Suppose that* $(\boldsymbol{E}^*, \boldsymbol{R}^*)$ *is a solution to the SPP that defines a competitive equilibrium with price process* $\boldsymbol{P}^e$*. Then,*

$$P^e(t) = \nabla v\,(E^*(t)) + \nabla c^{\mathsf{bo}}\,(D(t) - E^*(t)), \quad t \ge 0\,. \tag{1.32}$$

Proof. In the single-bus model we have $\mathcal{W}_\mathsf{D}(t){:=}v(E_\mathsf{D}(t)){-}c^{\mathsf{bo}}(D(t){-}E_\mathsf{D}(t)){-}P^e(t)E_\mathsf{D}(t)$. The result (1.32) follows because $\boldsymbol{E}^* = \boldsymbol{E}_\mathsf{D}$ in the competitive equilibrium, and the consumer is myopic (recall that the consumer does not consider ramp constraints). ☐

To find the average price, we consider the supplier's optimization problem. We then consider a Lagrangian relaxation, in which the constraint $E_\mathsf{S}(0) + R_\mathsf{S}(0) = g_0$ is captured by the Lagrange multiplier ν. The Lagrangian is denoted,

$$\mathcal{L}_\mathsf{S}(\boldsymbol{E}_\mathsf{S}, \boldsymbol{R}_\mathsf{S}, \nu) \;=\; \mathsf{E}\Big[\int e^{-\gamma t}\mathcal{W}_\mathsf{S}(t)\, dt\Big] - \nu[E_\mathsf{S}(0) + R_\mathsf{S}(0) - g_0]\,. \tag{1.33}$$

The following result is a consequence of the local Lagrange multiplier theorem [21, Theorem 1 of Sec. 9.3].

Lemma 2. *Let* $\mathsf{X}_{\mathsf{S}}^{g_0} \subset \mathsf{X}_{\mathsf{S}}$ *represent the set of feasible* $(\boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}})$*, subject to the given initial condition* g_0*. Suppose* $(\boldsymbol{E}_{\mathsf{S}}^{g_0}, \boldsymbol{R}_{\mathsf{S}}^{g_0})$ *maximizes* $\mathcal{L}_{\mathsf{S}}(\boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}}, 0)$ *over the set* $\mathsf{X}_{\mathsf{S}}^{g_0}$*. Then, there exist* $\nu^* \in \mathbb{R}$ *such that* $(\boldsymbol{E}_{\mathsf{S}}^{g_0}, \boldsymbol{R}_{\mathsf{S}}^{g_0})$ *maximizes* $\mathcal{L}_{\mathsf{S}}(\boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}}, \nu^*)$ *over the larger set* X_{S}*.* □

The next result is a construction required in the application of the local Lagrange multiplier result of Lemma 2.

Lemma 3. *Suppose that* $(\boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}}) \in \mathsf{X}_{\mathsf{S}}$*. Then there exists a family of solutions* $\{(\boldsymbol{E}_{\mathsf{S}}^{\alpha}, \boldsymbol{R}_{\mathsf{S}}) : |\alpha| \le 1\} \subset \mathsf{X}_{\mathsf{S}}$ *satisfying* $E_{\mathsf{S}}^{\alpha}(0) = \max(E_{\mathsf{S}}(0) + \alpha, 0)$*;* $|E_{\mathsf{S}}^{\alpha}(t) - E_{\mathsf{S}}(t)| \le |\alpha|$ *for all* t*; and,* $\lim_{\alpha \to 0} \frac{1}{\alpha}(E_{\mathsf{S}}^{\alpha}(t) - E_{\mathsf{S}}(t)) = \mathbf{1}_{\mathsf{S}}^{+}(t) := \mathbb{I}\{E_{\mathsf{S}}(t) > 0\}$ *for* $t > 0$*.* □

The extension of the average-price formula (1.31) is obtained on combining these results:

Theorem 4. *Suppose that* $(\boldsymbol{E}^*, \boldsymbol{R}^*)$ *is a solution to the SPP that defines a competitive equilibrium with price process* $\boldsymbol{P}^e$*. Suppose that* $E^*(0) > 0$*. Moreover, assume that the following strengthening of (A2) holds: The cost function satisfies* $c^{\mathsf{E}}(e) + |\nabla c^{\mathsf{E}}(e)| \le k_0(1 + e^2)$ *for some* $k_0 > 0$ *and for all* $e \ge 0$*. Then,*

$$\mathsf{E}\Big[\int_0^{\infty} e^{-\gamma t} \mathbf{1}_{\mathsf{S}}^{+}(t) P^e(t)\, dt\Big] \;=\; \mathsf{E}\Big[\int_0^{\infty} e^{-\gamma t} \mathbf{1}_{\mathsf{S}}^{+}(t) \nabla c^{\mathsf{E}}\left(E^*(t)\right) dt\Big] + \nu^*\,, \quad (1.34)$$

where ν^* *is the sensitivity term from Lemma 2.*

Proof. Under the assumptions of the theorem, the Lagrangian $\mathcal{L}_{\mathsf{S}}(\boldsymbol{E}^{\alpha}, \boldsymbol{R}^*, \nu)$ can be differentiated by α so that

$$\frac{d}{d\alpha}\mathcal{L}_{\mathsf{S}}(\boldsymbol{E}^{\alpha}, \boldsymbol{R}^*, \nu^*) = \mathsf{E}\Big[\int e^{-\gamma t} \frac{d}{d\alpha}\mathcal{W}_{\mathsf{S}}^{\alpha}(t)\, dt\Big] - \frac{d}{d\alpha}\nu^*[E^{\alpha}(0) + R^*(0) - g_0]\,, \quad (1.35)$$

where $\mathcal{W}_{\mathsf{S}}^{\alpha}(t) := P^e(t) E_{\mathsf{S}}^{\alpha}(t) - c^{\mathsf{E}}(E_{\mathsf{S}}^{\alpha}(t)) - c^{\mathsf{R}}(R^*(t))$. In this calculation, the square integrability assumption and bounds on c^{E} justify taking the derivative under the expectation and integral in (1.33).

Then, the result follows from two facts. First, the optimality of the Lagrangian at $\alpha = 0$ gives $\frac{d}{d\alpha}\mathcal{L}_{\mathsf{S}}(\boldsymbol{E}^{\alpha}, \boldsymbol{R}^*, \nu) = 0$ for $\alpha = 0$. Second, application of Lemma 3 followed by the chain rule gives,

$$\frac{d}{d\alpha}\mathcal{W}_{\mathsf{S}}^{\alpha}(t)\Big|_{\alpha=0} = \mathbf{1}_{\mathsf{S}}^{+}(t)\big(P^e(t)E^*(t) - \nabla c^{\mathsf{E}}\left(E^*(t)\right)\big)\,.$$

Evaluating (1.35) at $\alpha = 0$ then gives the desired result. □

Thus, we can establish an extended version of (1.31). The average price depends on the initial value g_0, and this dependence is captured through the sensitivity term ν^*. When ramping down is unconstrained, $\nu^* \ge 0$.

1.4.2 Network Model

We now study the network-based dynamic market model introduced in Sec. 1.2. In general, the analysis of this model is complicated by the constraints imposed on the transmission lines, in addition to the dynamic constraints on generation. Therefore, we look for appropriate relaxations to proceed with the analysis.

The relaxation introduced here is based on the assumption that the network and the consumer are not subject to dynamic constraints. Consequently, these two players are myopic in their respective optimization problems. On the other hand, the dynamic nature of the generation constraints restricts the supplier and, consequently, the social planner from adopting myopic strategies in their optimization problems.

To characterize the equilibrium prices, we introduce a market without any suppliers. It is a fictitious model introduced solely for analysis:

Definition 3. The $\$$-market is defined as follows:

(i) The models for the consumer and network players are unchanged. The supplier model is modified by relaxing the operational/physical constraints (1.3) on $(\boldsymbol{E}_{\mathsf{S}}, \boldsymbol{R}_{\mathsf{S}})$.

(ii) The welfare functions of the consumer and the network are unchanged. The welfare function of the supplier is *identically zero*. This is achieved by overriding the production cost functions as follows:

$$c_n^{\mathsf{E}\$}(E_{\mathsf{S}n}(t)) = P_n^e(t) E_{\mathsf{S}n}(t)\ , \quad c_n^{\mathsf{R}\$}(R_{\mathsf{S}n}(t)) = 0\ . \tag{1.36}$$

Since the welfare function $\mathcal{W}_{\mathsf{S}}^{\$}$ for the supplier is identically zero, the $\$$-market essentially reduces to a model consisting of two players: the consumer and the network. We find that the equilibrium for the original market provides an equilibrium for this two-player market.

Lemma 4. *A competitive equilibrium for the original three-player market is a competitive equilibrium for the two-player $\$$-market.*

Proof. Let $\{\boldsymbol{E}_{\mathsf{D}}^e, \boldsymbol{E}_{\mathsf{S}}^e, \boldsymbol{R}_{\mathsf{S}}^e, \boldsymbol{P}^e\}$ be a competitive equilibrium of the original three-player market. Clearly, the triple $\{\boldsymbol{E}_{\mathsf{D}}^e, \boldsymbol{E}_{\mathsf{S}}^e, \boldsymbol{R}_{\mathsf{S}}^e\}$ satisfies the supply-demand balance and the network constraints, and maximizes the consumer and the network welfare functions under price $\boldsymbol{P}^e$. Since the supplier's welfare $\mathcal{W}_{\mathsf{S}}^{\$}$ is always zero by assumption, we can view the pair $\{\boldsymbol{E}_{\mathsf{S}}^e, \boldsymbol{R}_{\mathsf{S}}^e\}$ as maximizing the supplier's welfare in the $\$$-market. Thus, the lemma holds. □

Therefore, *all* equilibrium prices $\boldsymbol{P}^e$ in the original market model support a competitive equilibrium in the $\$$-market. Hence, we can hope to extract properties of $\boldsymbol{P}^e$ by analyzing the simpler $\$$-market model. This is the main motivation behind the introduction of the $\$$-market.

Recall that in the single-bus model, the characterization of the price in (1.32) was based on the assumption that consumers are not subject to dynamic constraints and are hence myopic. Lemma 5, which follows from the definition of \$-market, justifies the same approach to analyze the network model.

Lemma 5. *All players, as well as the social planner, are myopic in the \$-market.* □

Hence the optimization problems of the consumer and the network in the \$-market are reduced to a "snapshot model" in which we can fix a time t to obtain properties of $P^e(t)$, exactly similar to the derivation of (1.32).

With time t fixed, we can suppress the time notation so that the snapshot version of the SPP is given as the maximization of the total welfare $\mathcal{W}^{\$}_{\mathsf{tot}}$ with

$$\mathcal{W}^{\$}_{\mathsf{tot}} \;=\; \sum_n \Big[v_n(E_{\mathsf{D}n}) - c_n^{\mathsf{bo}}(D_n - E_{\mathsf{D}n}) - P^e_n E_{\mathsf{S}n} \Big] \,, \tag{1.37}$$

subject to the following constraints:

$$\begin{aligned} 1^T E_{\mathsf{S}} = 1^T E_{\mathsf{D}} \qquad & \leftrightarrow \lambda \,, \\ -f_l^{\mathsf{max}} \le (E_{\mathsf{S}} - E_{\mathsf{D}})^T H_l \le f_l^{\mathsf{max}} \qquad & \leftrightarrow \mu_l^-, \mu_l^+ \ge 0 \qquad \text{for all } l \,, \\ 0 \le E_{\mathsf{D}n} \le D_n \qquad & \leftrightarrow \eta_n^-, \eta_n^+ \ge 0 \qquad \text{for all } n \,. \end{aligned}$$

$\lambda, \mu_l^-, \mu_l^+, \eta_n^-, \eta_n^+$ are Lagrange multipliers associated with the corresponding constraints. The Lagrangian of the SPP for the \$-market is the function of static variables:

$$\begin{aligned} \mathcal{L}^{\$} = & -\mathcal{W}^{\$}_{\mathsf{tot}} \;+\; \lambda(1^T(E_{\mathsf{D}} - E_{\mathsf{S}})) \;+\; \sum_l \mu_l^+[(E_{\mathsf{S}} - E_{\mathsf{D}})^T H_l - f_l^{\max}] \\ & + \sum_l \mu_l^-[-(E_{\mathsf{S}} - E_{\mathsf{D}})^T H_l - f_l^{\max}] \;+\; \sum_n \eta_n^+(E_{\mathsf{D}n} - D_n) - \sum_n \eta_n^+ E_{\mathsf{D}n} \,. \end{aligned} \tag{1.38}$$

For the fixed time t, we characterize the nodal price P^e_n as follows.

Proposition 2. *Consider the SPP for the \$-market with welfare function defined in* (1.37)*. Suppose that* μ_l^-, μ_l^+, η_n^-, η_n^+ *are the non-negative, optimal solutions to the dual with Lagrangian* (1.38)*. Then the equilibrium price has entries given as follows: For* $n = 1, \ldots, N$,

$$P^e_n = \nabla v_n(E^e_{\mathsf{D}n}) + \nabla c_n^{\mathsf{bo}}(D_n - E^e_{\mathsf{D}n}) + \Lambda \,, \tag{1.39}$$

where $\Lambda = \begin{cases} 0\,, & 0 < E^e_{\mathsf{D}n} < D_n \\ -\eta_n^+\,, & E^e_{\mathsf{D}n} = D_n \\ \eta_n^-\,, & E^e_{\mathsf{D}n} = 0 \end{cases}$,

and, $E^e_{\mathsf{D}n}$ *is the energy consumed in the equilibrium.*

Proof. Since $\{\boldsymbol{E}^e_{\mathsf{D}}, \boldsymbol{E}^e_{\mathsf{S}}, \boldsymbol{R}^e_{\mathsf{S}}, \boldsymbol{P}^e\}$ is a competitive equilibrium for the \$-market, $\{\boldsymbol{E}^e_{\mathsf{D}}, \boldsymbol{E}^e_{\mathsf{S}}\}$ maximizes the SPP for the \$-market. By the KKT conditions we obtain,

$$0 = \frac{\partial \mathcal{L}^{\$}}{\partial E_{\mathsf{D}n}} = -\nabla v_n(E^e_{\mathsf{D}n}) - \nabla c^{\mathsf{bo}}_n(D_n - E^e_{\mathsf{D}n}) + \lambda \quad + \sum_l (\mu^-_{ln} - \mu^+_{ln}) H_{ln} + \eta^+_n - \eta^-_n \; ;$$

$$0 = \frac{\partial \mathcal{L}^{\$}}{\partial E_{sn}} = P^e_n - \lambda - \sum_l (\mu^-_{ln} - \mu^+_{ln}) H_{ln} \; .$$

On summing these two equations we obtain,

$$P^e_n = \nabla v_n(E^e_{\mathsf{D}n}) + \nabla c^{\mathsf{bo}}_n(D_n - E^e_{\mathsf{D}n}) - \eta^+_n + \eta^-_n \; .$$

The proposition follows from the complementary slackness conditions. □

Note that result (1.39) holds for all equilibria in the two-player \$-market. Consequently, from Lemma 4, it holds for all equilibria in the original three-player market. Although it appears from (1.39) that prices depend upon the actions of the players, this is not the case. Similar to the Prop. 1, we can write the price as

$$P^e_n(t) = \nabla v_n(E^*_{\mathsf{D}n}(t)) + \nabla c^{\mathsf{bo}}_n(D_n(t) - E^*_{\mathsf{D}n}(t)) + \Lambda^*(t) \; ,$$

where $\{E^*_{\mathsf{D}n}(t)\}$ and, consequently, $\Lambda^*(t))$ are obtained from the solution of the SPP.

To apply Prop. 2, we must identify the parameters $\{\eta^-_n, \eta^+_n\}$, which represent the sensitivities of the SPP solution with respect to the constraints $E_{\mathsf{D}n} \geq 0$ and $E_{\mathsf{D}n} \leq D_n$ respectively. Next, we motivate the price computations through numerical examples.

1.4.3 Texas Model: Range of Prices

We revisit Example B to illustrate the application of Prop. 2. We show how prices in a network with binding transmission constraints can go below zero or well above the choke-up price.

For the Texas model shown in Fig. 1.2, we assume that a single supplier is located at bus 1 while buses 2 and 3 are load buses. The buyer has linear utility of consumption and disutility of blackout (1.1b) at buses 2 and 3, with identical parameters v and c^{bo}. With identical impedances for the three transmission lines and bus 1 chosen as the reference bus, the injection shift factor matrix H is given by the expression (1.13).

We assume that the supporting price P_1^e at bus 1 is *zero*, and that this is true not only for the snapshot values $\{E_{\mathsf{D}i}, E_{\mathsf{S}i}, R_{\mathsf{S}i}\}$ and parameters $\{f_{ij}^{\max}\}$, but also for all values in a neighborhood of these nominal values. This is not unreasonable based on the results of Sec. 1.4.1 if the reserves are strictly positive at bus 1. Under these assumptions we can then compute the prices P_2^e and P_3^e at buses 2 and 3 respectively, using the $\$$-market introduced in Sec. 1.4.2.

Recall that the supplier's welfare function is identically zero in the $\$$-market. Suppose that $D_2 = 170$ MW and $D_3 = 30$ MW. Then, the snapshot SPP for the $\$$-market is,

$$\begin{aligned} \min \quad & -\big[v(E_{\mathsf{D}2} + E_{\mathsf{D}3}) - c^{\mathsf{bo}}(200 - E_{\mathsf{D}2} - E_{\mathsf{D}3})\big] \\ \text{subject to} \quad & E_{\mathsf{S}1} = E_{\mathsf{D}2} + E_{\mathsf{D}3} \\ & -f_{12}^{\max} \le \tfrac{2}{3}E_{\mathsf{D}2} + \tfrac{1}{3}E_{\mathsf{D}3} \le f_{12}^{\max} \\ & -f_{23}^{\max} \le \tfrac{1}{3}E_{\mathsf{D}2} - \tfrac{1}{3}E_{\mathsf{D}3} \le f_{23}^{\max} \\ & -f_{13}^{\max} \le \tfrac{1}{3}E_{\mathsf{D}2} + \tfrac{2}{3}E_{\mathsf{D}3} \le f_{13}^{\max} \\ & 0 \le E_{\mathsf{D}2} \le 170\,,\ 0 \le E_{\mathsf{D}3} \le 30\ . \end{aligned}$$

Negative prices: Assume that $f_{23}^{\max} = 40$ MW, while the other two lines are unconstrained. Solving the SPP for the $\$$-market we obtain $E_{\mathsf{D}2} = 150$ MW and $E_{\mathsf{D}3} = 30$ MW, and we find that the limit $f_{23}^{\max} = 40$ MW is reached. Since $0 < E_{\mathsf{D}2} < D_2$, we have $P_2^e = v + c^{\mathsf{bo}}$ by Prop. 2.

For a given $\epsilon \in \mathbb{R}$, we perturb the constraint on $E_{\mathsf{D}3}$ to obtain $0 \le E_{\mathsf{D}3} \le 30 + \epsilon$. On re-solving the SPP we obtain $E_{\mathsf{D}2} = 150 + \epsilon$ MW and $E_{\mathsf{D}3} = 30 + \epsilon$ MW. Applying Prop. 2, P_3^e is given by the limit,

$$P_3^e \;:=\; v + c^{\mathsf{bo}} + \lim_{\epsilon\to 0} \frac{-(180 + 2\epsilon)v + (20 - 2\epsilon)c^{\mathsf{bo}} + 180v - 20c^{\mathsf{bo}}}{\epsilon}\ .$$

That is, $P_3^e = -(v + c^{\mathsf{bo}})$, which is clearly *negative.*

Prices exceeding the choke up price: Assume that $f_{13}^{\max} = 50$ MW, while the other two lines are unconstrained. Again, solving the SPP for the $\$$-market gives $E_{\mathsf{D}2} = 150$ MW and $E_{\mathsf{D}3} = 0$ MW with the limit $f_{13}^{\max} = 50$ MW being reached. Prop. 2 gives $P_2^e = v + c^{\mathsf{bo}}$ since $0 < E_{\mathsf{D}2} < D_2$.

For a given $\epsilon \in \mathbb{R}$, we perturb the constraint on $E_{\mathsf{D}3}$ to obtain $0+\epsilon \le E_{\mathsf{D}3} \le 30$. On re-solving the SPP, we obtain $E_{\mathsf{D}2} = 150 - 2\epsilon$ MW and $E_{\mathsf{D}3} = \epsilon$ MW. Again, applying Prop. 2, P_3^e is expressed as a limit,

$$P_3^e \;:=\; v + c^{\mathsf{bo}} + \lim_{\epsilon\to 0} \frac{-(180 - \epsilon)v + (20 + \epsilon)c^{\mathsf{bo}} + 180v - 20c^{\mathsf{bo}}}{\epsilon}\ .$$

That is, $P_3^e = 2(v + c^{\mathsf{bo}})$, which is twice the choke-up price.

Thus, when the transmission constraints come into play, the equilibrium prices are spread over a wide range, encompassing values well below zero as well as values far exceeding the choke-up price.

1.4.4 Ancillary Service Prices

Given the volatile nature of prices in the dynamic setting, we are interested in understanding the implications of volatility on the providers of ancillary service. The results presented in the preceding sections can be adapted to a market setting in which a number of services can be used to meet the demand.

Here we restrict to the model of [6] presented in Example C of Sec. 1.2.2. We consider two sources of generation − primary and ancillary − whose instantaneous output at time t is denoted by $G_{\mathsf{p}}(t)$ and $G_{\mathsf{a}}(t)$ respectively. For simplicity, we focus on the case in which ramping down is unconstrained, i.e., $\zeta_{\mathsf{p}}^-, \zeta_{\mathsf{a}}^- = \infty$.

We assume that the consumers exhibit *demand-response* capabilities: certain loads can be turned off to maintain supply-demand balance in the event of reserves shortfall. When the demand exceeds the available supply, loads with demand response capabilities are the first ones to be turned off. If the load of the responsive consumers can sufficiently cover the reserve shortfall, the price is set by the cost of demand response. Otherwise, forced blackout occurs and the price equals the choke-up price. Since the cost of demand response is typically lower than the cost of blackout, the price when the supply deficiency can be covered by demand response is lower than that if demand response was unavailable. Thus, demand response acts as a cushion between the normal secure operations and the blackout. We refer the reader to [20] and the references therein for more details on demand response.

In our model, we use $\bar{r}^{\mathsf{dr}}_{\mathsf{max}}$ to denote total load of the consumers with demand response capability. That is, we have forced blackouts only if $R(t) \leq -\bar{r}^{\mathsf{dr}}_{\mathsf{max}}$. In this example, we convert the SPP into a cost minimization problem, in which the cost function on $(\boldsymbol{G}^{\mathsf{p}}, \boldsymbol{G}^{\mathsf{a}}, \boldsymbol{R})$ has the following form: For $t \geq 0$,

$$\mathcal{C}(t) \ := \ c^{\mathsf{p}} G_{\mathsf{p}}(t) + c^{\mathsf{a}} G_{\mathsf{a}}(t) + (c^{\mathsf{bo}} - c^{\mathsf{dr}})\mathbb{I}(R(t) < -\bar{r}^{\mathsf{dr}}_{\mathsf{max}}) + c^{\mathsf{dr}}\mathbb{I}(R(t) < 0)$$

where c^{p}, c^{a} represent the per unit production costs of primary and ancillary services respectively, c^{dr} is the cost of demand response-based load shedding and c^{bo} is the cost of blackout [6].

To investigate the impact of prices on the ancillary service providers, we simulate the market based on a controlled random-walk (CRW) model for demand, where $D(t)$ has the form,

$$D(k+1) = D(k) + \mathcal{E}(k+1), \qquad k \geq 0, \quad D(0) = 0 \,,$$

in which the increment process $\boldsymbol{\mathcal{E}}$ is a discrete-valued, bounded, i.i.d. sequence, with marginal distribution symmetric on $\{\pm 1\}$. The reserve $\boldsymbol{R}$ is modeled in discrete time using the expression (1.15). The other parameters used are as follows: $c^{\mathsf{p}} = 1$, $c^{\mathsf{a}} = 20$, $c^{\mathsf{dr}} = 100$ and $c^{\mathsf{bo}} = 400$. The ramp-up rates are taken as $\zeta_{\mathsf{p}}^{+} = 1/10$ and $\zeta_{\mathsf{a}}^{+} = 2/5$.

We describe results from experiments using threshold policies, based on two thresholds $(\bar{r}^{\mathsf{p}}, \bar{r}^{\mathsf{a}})$. Under the policy considered, primary service is ramped up whenever $R(t) \leq \bar{r}^{\mathsf{p}}$. Similarly, ancillary service is ramped up whenever $R(t) \leq \bar{r}^{\mathsf{a}}$. The simulation experiments from [5] were conducted in discrete-time: The discount factor $\beta = 0.995$ used there corresponds approximately to the discount rate $\gamma = 0.005$ in (1.21). We refer the reader to [5] further details about the threshold policy.

We find the "best-threshold" by approximating the discounted cost by the standard Monte-Carlo estimate. We use c^{dr} and $\bar{r}^{\mathsf{dr}}_{\mathsf{max}}$ as simulation parameters to study sensitivity of average prices to demand response capabilities.

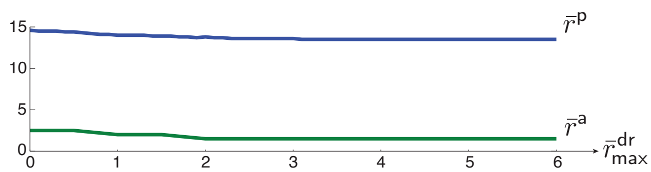

Fig. 1.4 Optimal thresholds for primary and ancillary service

In Fig. 1.4, we plot the optimal thresholds for primary and ancillary service against the demand response capacity $\bar{r}^{\mathsf{dr}}_{\mathsf{max}}$. Note that the optimal threshold for primary service is much higher than that of ancillary service, since primary service ramps up slower than ancillary service and is less expensive. The low sensitivity is consistent with the conclusions of [6].

The average price for primary service, and the conditional average price for ancillary service are shown in Fig. 1.5, for different values of $\bar{r}^{\mathsf{dr}}_{\mathsf{max}}$. The average price $\mathsf{E}[P^e] = c^{\mathsf{p}}$ is consistent with the conclusions of [7] and Theorem 4 (recall that G_{p} is not sign-constrained). The conditional average ancillary service price is given by

$$\mathsf{E}[P^e | G_{\mathsf{a}} > 0] = \Big[\int_0^\infty e^{-\gamma t} P^e(t) \mathbb{I}(G_{\mathsf{a}}(t) > 0)\, dt \Big] \Big[\int_0^\infty e^{-\gamma t} \mathbb{I}(G_{\mathsf{a}}(t) > 0)\, dt \Big]^{-1}.$$

The results shown are consistent with (1.34): It appears that when $\bar{r}^{\mathsf{dr}}_{\mathsf{max}} \geq 3$, $\nu^* \approx 5 \Big(\int_0^\infty e^{-\gamma t} \mathbb{I}(G_{\mathsf{a}}(t) > 0)\, dt \Big)$.

Also shown in Fig. 1.5 is a plot of the variance of the equilibrium price with respect to $\bar{r}^{\mathsf{dr}}_{\mathsf{max}}$. We see that the price variance drops dramatically with an increase in the demand response capacity, even though the optimal reserve thresholds are virtually unchanged.

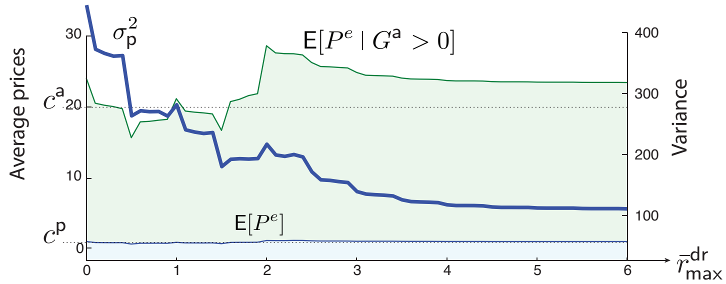

Fig. 1.5 Average prices and variance of P^e.

Recall that the prices for primary and ancillary service are identical. However, the bulk of primary service is allocated in the day ahead market. Consequently, ancillary service, such as provided by gas turbines, will be exposed to much greater variability in the efficient market outcome. We hypothesize that high risk and low average prices may drive generators out of business. Market operators and regulators should consider such consequences when designing electricity markets.

1.5 Conclusions

In this chapter, we have described a framework for constructing dynamic models for electricity markets, and methods for characterizing the resulting competitive equilibria. The dynamic model is constructed using techniques well-known in the control community and can effectively handle the underlying physics of the power system while taking into account the economic aspects of electricity trading.

Competitive equilibrium theory has elegant mathematical underpinnings, and gives some insight into market outcomes. For example, volatile electricity prices seen throughout the world today are not surprising, given the theory surveyed in Sec. 1.4.

However, just as in equilibrium theory for control (as taught in senior-level undergraduate electrical engineering courses), an equilibrium is an extreme idealization, intended merely as a *starting point* for understanding market behavior. In particular, (A1) is rarely strictly valid: consumers and suppliers do not share the same information. The price taking assumption in (A3) does not hold in real-life energy markets. The quantitative formulation of “cost” in the SPP requires more careful consideration to capture a broader range of issues: In our discussions we have ignored environmental impact and other long-term costs [11].

We urgently need to think about market design as engineers think about control design: How can we formulate market rules that assure reliability on a range of time-scales, provide incentives for new technology that is inexpensive in terms of cost and environmental impact, and adapt to an evolving environment and populace?

We hope that the ideas described in this chapter will form building blocks for addressing these questions; our ambition is to create a richer science for the design of smarter energy networks.

Acknowledgements Research supported in part by the Grainger Endowments to the University of Illinois, DOE awards DE-OE0000097 (TCIPG) and DE-SC0003879, and AFOSR grant FA9550-09-1-0190.

References

1. C. D. Aliprantis, B. Cornet, and R. Tourky, *Economic equilibrium: Optimality and price decentralization*, Positivity **6** (2002), 205–241.
2. E. J. Anderson and A. B. Philpott, *Using supply functions for offering generation into an electricity market*, Oper. Res. **50** (2002), no. 3, 477–489. MR 1910284 (2003f:91069)
3. K. J Arrow and F. Hahn, *General competitive analysis*, Holden-Day San Francisco, 1971.
4. R. Baldick, R. Grant, and E. Kahn, *Theory and application of linear supply function equilibrium in electricity markets*, Journal of Regulatory Economics **25** (2004), no. 2, 143–167.
5. M. Chen, I.-K. Cho, and S.P. Meyn, *Reliability by design in a distributed power transmission network*, Automatica **42** (2006), 1267–1281, (invited).
6. I.-K. Cho and S. P. Meyn, *A dynamic newsboy model for optimal reserve management in electricity markets*, Submitted for publication, SIAM J. Control and Optimization., 2009.
7. I.-K. Cho and S. P. Meyn, *Efficiency and marginal cost pricing in dynamic competitive markets with friction*, Theoretical Economics **5** (2010), no. 2, 215–239.
8. G. C. Chow, *Dynamic economics: Optimization by the lagrange method*, Oxford University Press, USA, 1997.
9. J.H. Chow, W. De Mello, and K.W. Cheung, *Electricity market design: An integrated approach to reliability assurance*, Proceedings of the IEEE **93** (2005), no. 11, 1956 –1969.
10. C. J. Day, B. F. Hobbs, and J.-S. Pang, *Oligopolistic competition in power networks: A conjectured supply function approach*, IEEE Transactions on Power Systems **17** (2002), 597–607.
11. P. R. Epstein, J. J. Buonocore, K. Eckerle, M. Hendryx, B. M. Stout III, R. Heinberg, R. W. Clapp, B. May, N. L. Reinhart, M. M. Ahern, S. K. Doshi, and L. Glustrom. Full cost accounting for the life cycle of coal. *Annals of the NY Academy of Sciences*, 1219(1):73–98, 2011.
12. FERC, *Staff Report to the Federal Energy Regulatory Commission on the Causes of Wholesale Electric Pricing Abnormalities in the Midwest During June 1998*, 1998, FERC study, September 22, 1998.
13. B. F. Hobbs, *Linear complementarity models of nash-cournot competition in bilateral and POOLCO power markets*, IEEE Transactions on Power Systems **16** (2001), no. 2, 194–202.
14. B. F. Hobbs and U. Helman, *Complementarity-based equilibrium modeling for electric power markets*, Modeling Prices in Competitive Electricity Markets, Wiley Series in Financial Economics, John Wiley & Sons Ltd, West Sussex, England, 2004, pp. 69–98.

15. Houston Chronicle Editorial, *Rolling blunder statewide blackouts raise the question: Who really runs the lone star power grid?*, www.chron.com/disp/story.mpl/editorial/7424733.html, Feb. 12 2011.
16. X. Hu and D. Ralph, *Using EPECs to model bilevel games in restructured electricity markets with locational prices*, Oper. Res. **55** (2007), no. 5, 809–827. MR MR2360950
17. R. Johari and J. N. Tsitsiklis, *Efficiency of scalar-parameterized mechanisms*, Operations Res. **57** (2009), 823–839.
18. A. C. Kizilkale and S. Mannor, *Volatility and efficiency in markets with friction*, 48th Annual Allerton Conference on Communication, Control, and Computing, 2010, pp. 50–57.
19. P. D. Klemperer and M. A. Meyer, *Supply function equilibria in oligopoly under uncertainty*, Econometrica **57** (1989), 1243–1277.
20. A. Kowli and S. P. Meyn, *Supporting wind generation deployment with demand response*, IEEE PES 11: Power Energy Society General Meeting, 2011.
21. D. G. Luenberger, *Optimization by vector space methods*, John Wiley, 1969.
22. A. Mas-Colell, M. D Whinston, and J. R. Green, *Microeconomic theory*, Oxford university press New York, 1995.
23. C. Metzler, B. Hobbs, and J-S. Pang, *Nash-cournot equilibria in power markets on a linearized DC network with arbitrage: Formulations and properties*, Networks & Spatial Economics **3** (2003), no. 2, 123 – 150.
24. S. Meyn, M. Negrete-Pincetic, G. Wang, A. Kowli, and E. Shafieepoorfard, *The value of volatile resources in electricity markets*, Proc. of the 49th IEEE Conf. on Dec. and Control, 2010, pp. 1029 –1036.
25. R. Mookherjee, T. Friesz, B. F. Hobbs, and M. Ringdon, *Dynamic oligopolistic competition on an electric power network with ramping costs and joint sales constraints*, Journal of Industrial Management and Organization **4** (2008), no. 3, 425–452.
26. M. Negrete-Pincetic and S.P. Meyn, *Intelligence by design for the entropic grid*, IEEE PES 11: Power Energy Society General Meeting, 2011, Invited lecture for PES panel: Deploying Tomorrow's Electric Power Systems: Low Carbon, Efficiency and Security.
27. S. Robinson, *Math model explains volatile prices in power markets*, SIAM News (2005).
28. U. V. Shanbhag, G. Infanger, and P. W. Glynn, *A complementarity framework for forward contracting under uncertainty*, To appear in Operations Research (2011).
29. J.C. Smith, S. Beuning, H. Durrwachter, E. Ela, D. Hawkins, B. Kirby, W. Lasher, J. Lowell, K. Porter, K. Schuyler, and P. Sotkiewicz, *The wind at our backs*, Power and Energy Magazine, IEEE **8** (2010), no. 5, 63 –71.
30. G. Wang, A. Kowli, M. Negrete-Pincetic, E. Shafieepoorfard, and S. Meyn, *A control theorist's perspective on dynamic competitive equilibria in electricity markets*, Proc. 18th World Congress of the International Federation of Automatic Control (IFAC) (Milano, Italy), 2011.
31. G. Wang, M. Negrete-Pincetic, A. Kowli, E. Shafieepoorfard, S. Meyn, and U. V. Shanbhag, *Real-time prices in an entropic grid*, Submitted to 2011 IEEE CDC, 2011.
32. *Electricity Authority strikes down Genesis price spike*, Business Desk Report, New Zealand Herald, May 6, 2011.
`www.nzherald.co.nz/business/news/ article.cfm?c_id=3&objectid=10723856`,
33. R. Wilson, *Architecture of power markets*, Econometrica **70** (2002), no. 4, 1299–1340.
34. A. J. Wood and B. F. Wollenberg, *Power generation, operation, and control*, Wiley, New York, 1996.
35. J. Yao, I. Adler, and S. Oren, *Modeling and computing two-settlement oligopolistic equilibrium in a congested electricity network*, Oper. Res. **56** (2008), no. 1, 34–47. MR MR2402216
36. V. M. Zavala and M. Anitescu. On the dynamic stability of electricity markets. Mathematical Programming, Submitted, 2010.