Finding the best mismatched detector for channel coding and hypothesis testing

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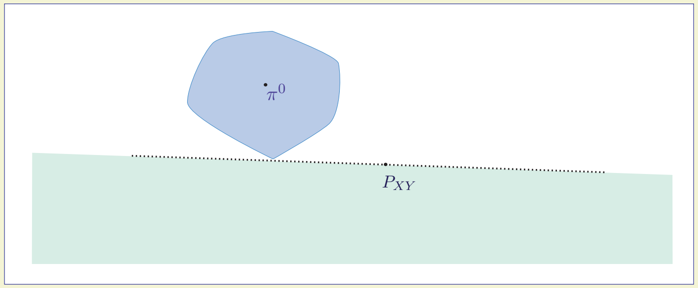
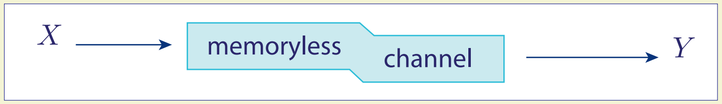
Outline

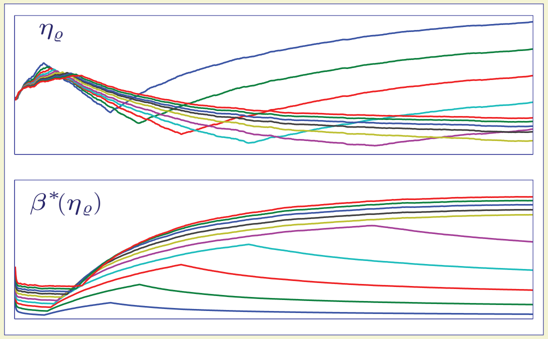
Introduction



Mismatched channel



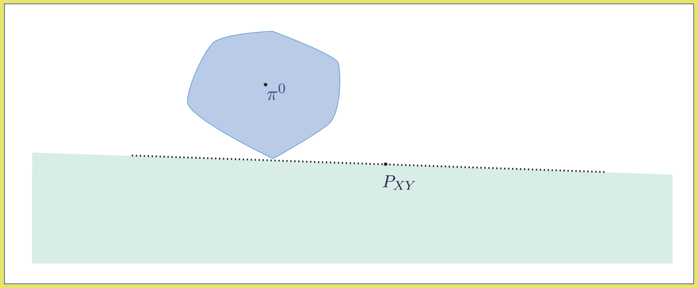
Learning the best detector



Conclusions

I

Introduction



Background: Mismatch Channel

$X \longrightarrow$ $\boxed{\text{memoryless channel}}$ $\longrightarrow Y$

Optimal detector ML: Based on the empirical distributions,

$$i^* = \arg\max_i \langle \Gamma_n^i, F \rangle$$

where, F is the log-likelihood ratio (real-valued function of x, y)

Γ_n^i is the empirical distribution of (X^i, Y)

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Question considered by Lapidoth 1994, Csiszar et. al. 1981, 1995:

What if the channel model is not known exactly ?

Background: Robust Hypothesis Testing

Uncertainty classes defined by moment constraints $\mathbb{P}_0$, $\mathbb{P}_1$

For given false alarm exponent η,
min-max optimal missed detection exponent β^*

$$\beta^* = \inf_{\pi_1 \in \mathbb{P}_1} \inf_{\mu \in \mathcal{Q}_\eta(\mathbb{P}_0)} D(\mu \,\|\, \pi_1)$$

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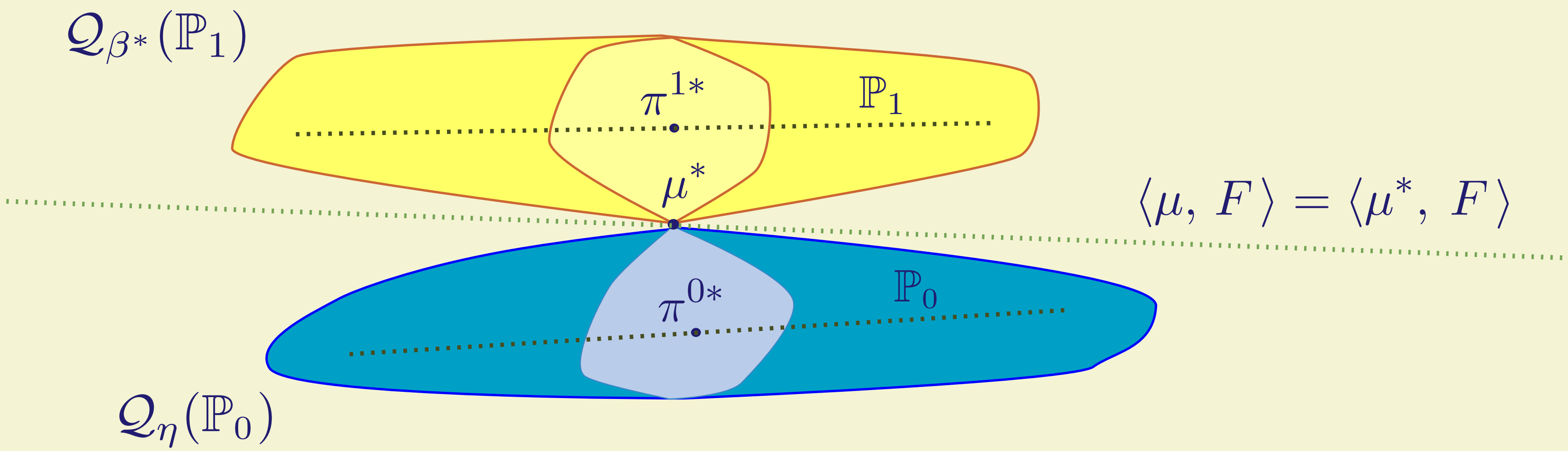

F : Generalized Log Likelihood Ratio

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Goal: Robustness

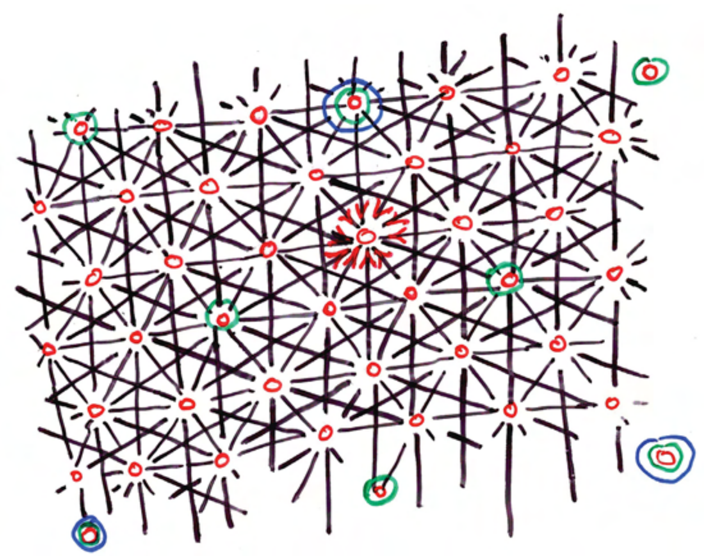
Original motivation for the research on mismatched detectors

What is the impact on capacity given a poor model?
How should codes be constructed
to take into account uncertainty?

Goal: Complexity

Detection and capacity - easy. *Its just mutual information*!!

Linear detectors suggest relaxation techniques for multiuser and network settings



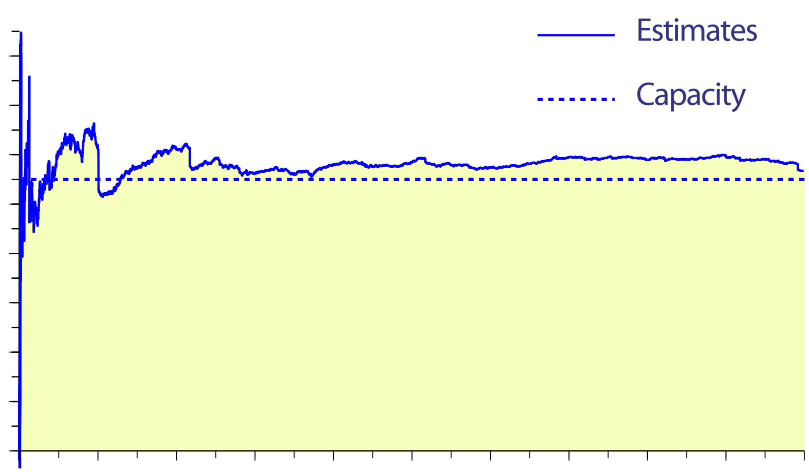
Goal: Adaptation

A new point of view!

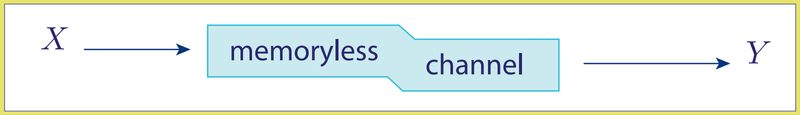
How can a detector be tuned on-line
in a dynamic environment?

$X \longrightarrow$ memoryless channel $\longrightarrow Y$

This will depend on SNR
Other users
Network conditions ...



II
Mismatched Channel



Mismatched Detector

A given function $F(x,y)$ defines a surrogate ML detector:

$$i^* = \arg\max_i \langle \Gamma_n^i, F \rangle = \arg\max_i \Big\{ \sum_{t=1}^{n} F(X_t^i, Y_t) \Big\}$$

Empirical distributions for ith codeword:

$$\Gamma_n^i = n^{-1} \sum_{t=1}^{n} \delta_{X_t^i, Y_t}$$

Reliably received rate (in general a lower bound):

Generalized Mutual Information (Lapidoth, Csiszar)

Mismatched Detector: Sanov's Theorem

Sanov's Theorem (in form of Chernoff's Bound) gives

$$\log P_e \leq -nI_{\mathrm{LDP}}(P_X; F)$$ (P_e *pair-wise prob. of error*)

$$I_{\mathrm{LDP}}(P_X; F) := \inf\Big\{D(\Gamma \| P_X \otimes P_Y) : \langle \Gamma, F \rangle = \langle P_{XY}, F \rangle\Big\}$$

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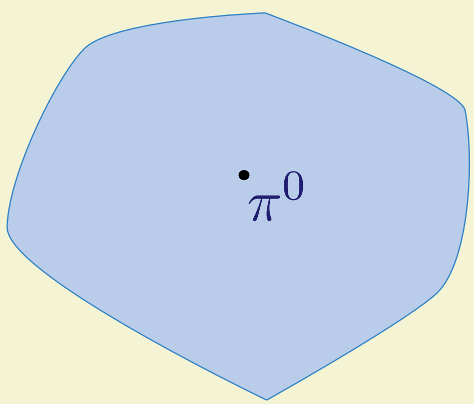

$$\mathcal{Q}_{\beta_0}(\pi^0) = \{ \Gamma : D(\Gamma \,\|\, \pi^0) \leq \beta_0 \}$$

$$\pi^0 = P_X \otimes P_Y$$

Mismatched Detector: Sanov's Theorem

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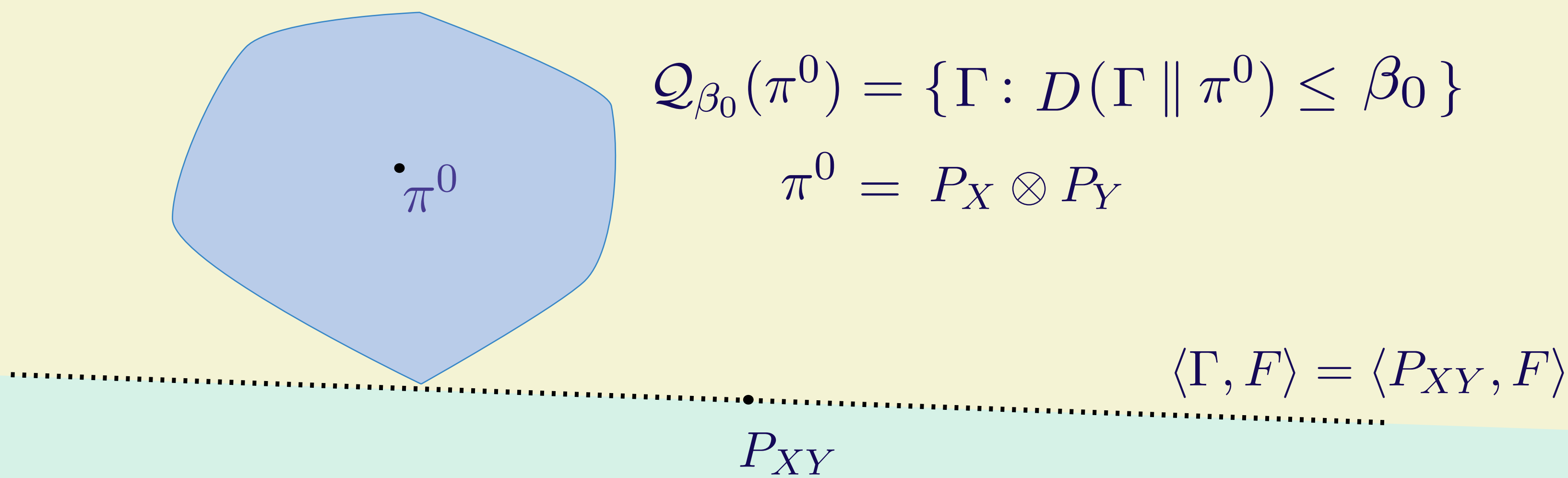

$$\beta_0 = I_{\mathrm{LDP}}(P_X; F)$$

Mismatched Detector: Generalized Mutual Information

Generalized Mutual Information - introduces typicality constraint,

$$\boxed{\log P_e \leq -nI_{\mathrm{GMI}}(P_X;F)}$$

$$I_{\mathrm{GMI}}(P_X;F) := \inf\Big\{D(\Gamma\|P_X\otimes P_Y) : \langle\Gamma,F\rangle = \langle P_{XY},F\rangle\,,\,\Gamma_2 = P_Y\Big\}$$

Mismatched Detector: Generalized Mutual Information

Generalized Mutual Information

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Derivation of GMI bound using Sanov:

Mismatched Detector: Generalized Mutual Information

Generalized Mutual Information

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$$I_{\mathrm{GMI}}(P_X; F) := \inf\Big\{D(\Gamma \| P_X \otimes P_Y) : \langle \Gamma, F \rangle = \langle P_{XY}, F \rangle \,,\, \Gamma_2 = P_Y\Big\}$$

Derivation of GMI bound using Sanov: For any function $G(y)$,

$$i^* = \arg\max_i \langle \Gamma_n^i, F + G \rangle = \arg\max_i \Big\{ \sum_{t=1}^{n} [F(X_t^i, Y_t) + G(Y_t)] \Big\}$$

$$\log P_e \leq -nI_{\mathrm{LDP}}(P_X; F + G) \quad \textit{any } G(y)$$

Mismatched Detector: Generalized Mutual Information

Generalized Mutual Information

$$\log P_e \leq -n I_{\mathrm{GMI}}(P_X; F)$$

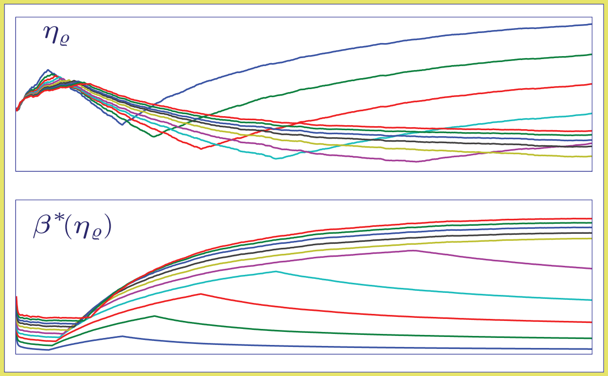
$$I_{\mathrm{GMI}}(P_X; F) := \inf\Big\{D(\Gamma \| P_X \otimes P_Y) : \langle \Gamma, F \rangle = \langle P_{XY}, F \rangle \,,\, \Gamma_2 = P_Y \Big\}$$

$$\log P_e \leq -n I_{\mathrm{LDP}}(P_X; F + G\,)$$ *any* $G(y)$

$$\sup_{G(y)} I_{\mathrm{LDP}}(P_X; F + G) = I_{\mathrm{GMI}}(P_X; F)$$

G^* : Lagrange multiplier for equality constraint $\Gamma_2 = P_Y$

III
Learning the Best Detector



Learning the Best Test

Given a parameterized class of detectors,

$$\{F_\alpha = \sum \alpha_i \psi_i : \alpha \in \mathbb{R}^d\}$$

What is the best value of α?

Maximize
MM capacity: $J(\alpha) := I_{\mathrm{LDP}}(P_X\,; F_\alpha)$

Learning the Best Test

Maximize

MM capacity: $J(\alpha) := I_{\mathrm{LDP}}(P_X; F_\alpha)$

Derivatives: $\nabla J = \bar{\psi} - \langle \Gamma^\alpha, \psi \rangle$

$$\nabla^2 J = \langle \Gamma^\alpha, \psi\psi^T \rangle - \langle \Gamma^\alpha, \psi \rangle \langle \Gamma^\alpha, \psi \rangle^T$$

where for any set $A \in \mathcal{B}(\mathsf{X} \times \mathsf{Y})$,

$$\Gamma^\alpha(A) := \frac{\langle P_X \otimes P_Y, e^{F_\alpha} \mathbb{I}_A \rangle}{\langle P_X \otimes P_Y, e^{F_\alpha} \rangle}$$

Learning the Best Test

Maximize
MM capacity: $J(\alpha) := I_{\mathrm{LDP}}(P_X; F_\alpha)$

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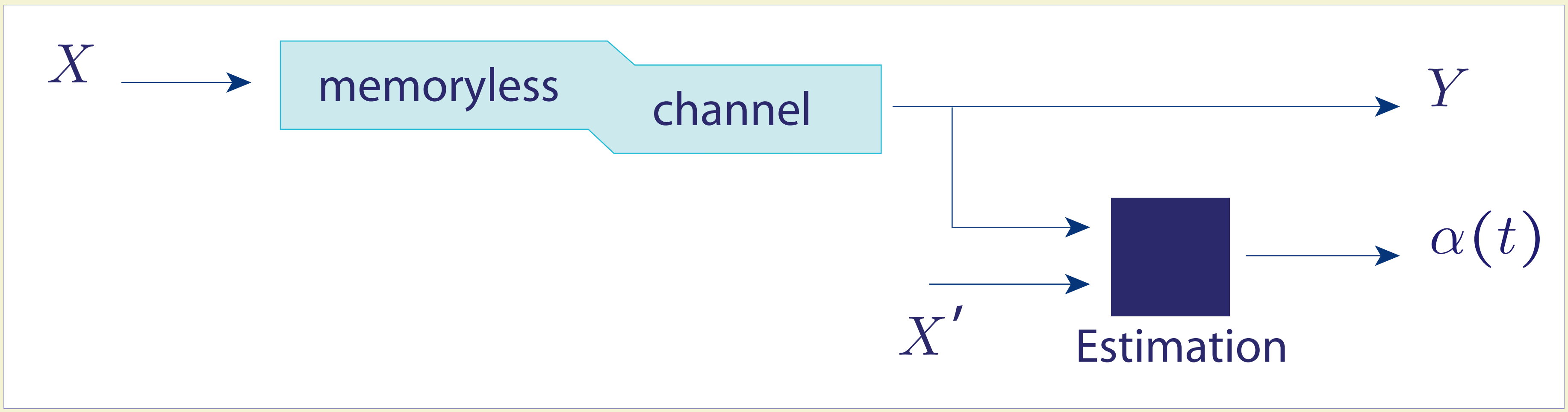
Newton Raphson based on channel model, or

Stochastic Newton Raphson based on measurements alone!

Learning the Best Test

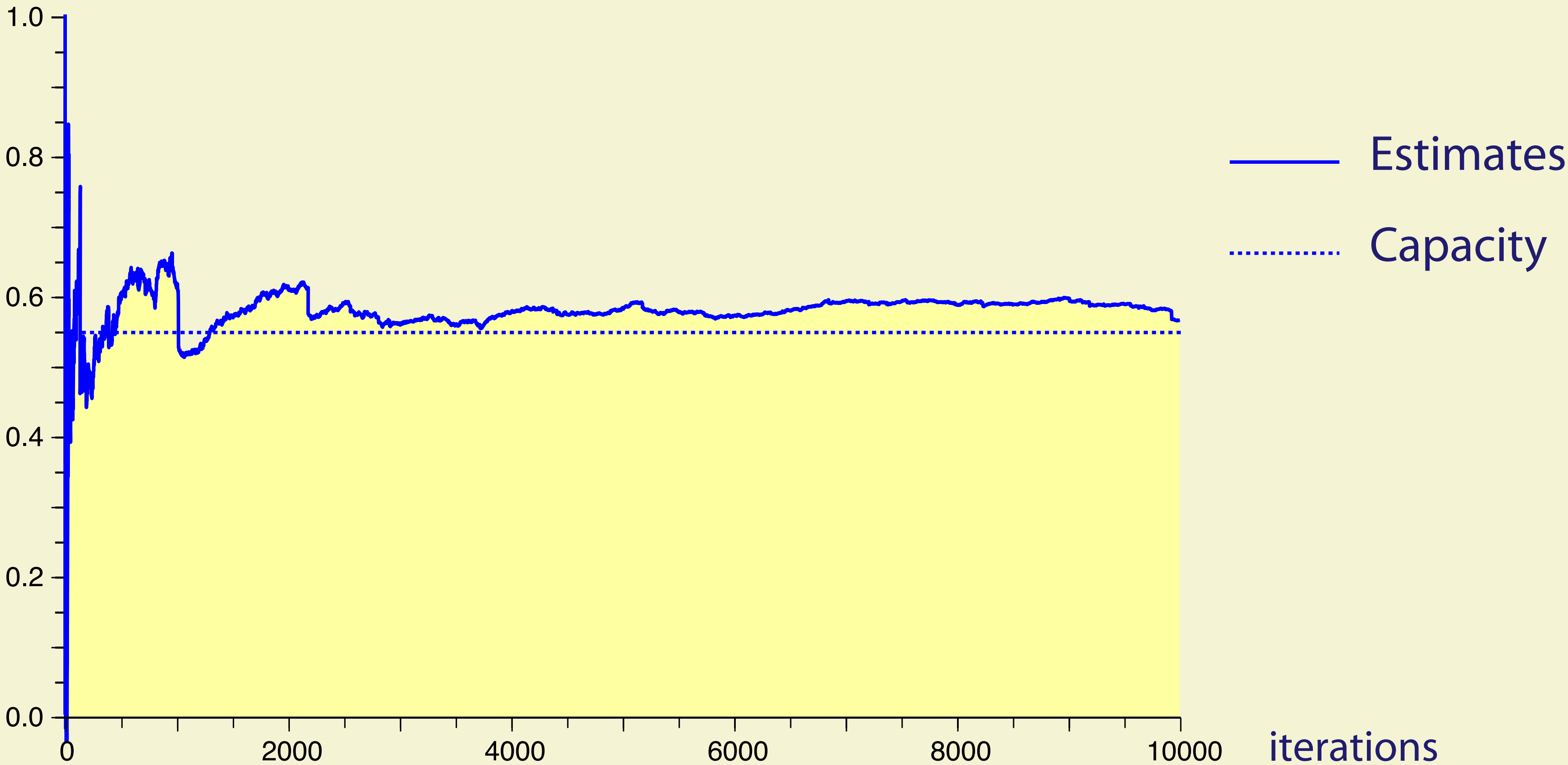
Estimation algorithm is *decentralized* and *model-free*



AWGN Channel

$Y = X + N$ SNR = 2

Basis: $\psi = (x^2, y^2, xy)^{\mathrm{T}}$



Mismatched Neyman Pearson Hypothesis Testing

Numerical experiment: X^1 uniform on $[0,1]$, $X^0 = \sqrt[5]{X^1}$

η *False Alarm Error Exponent*

$\beta^*(\eta)$ *Missed Detection Error Exponent*

Stochastic Newton-Raphson to find [η, $\beta^*(\eta)$] pairs

Parameterized class of detectors,

$$\{F_\alpha = \sum \alpha_i \psi_i : \alpha \in \mathbb{R}^d\}$$

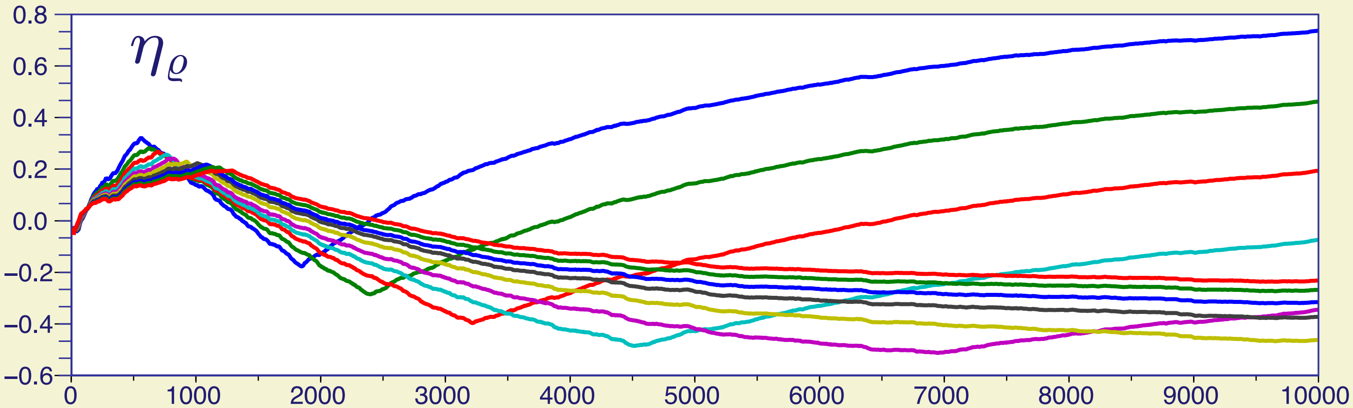
$$\psi(x) = [x, \log(x)/10]^T$$

Optimal detector

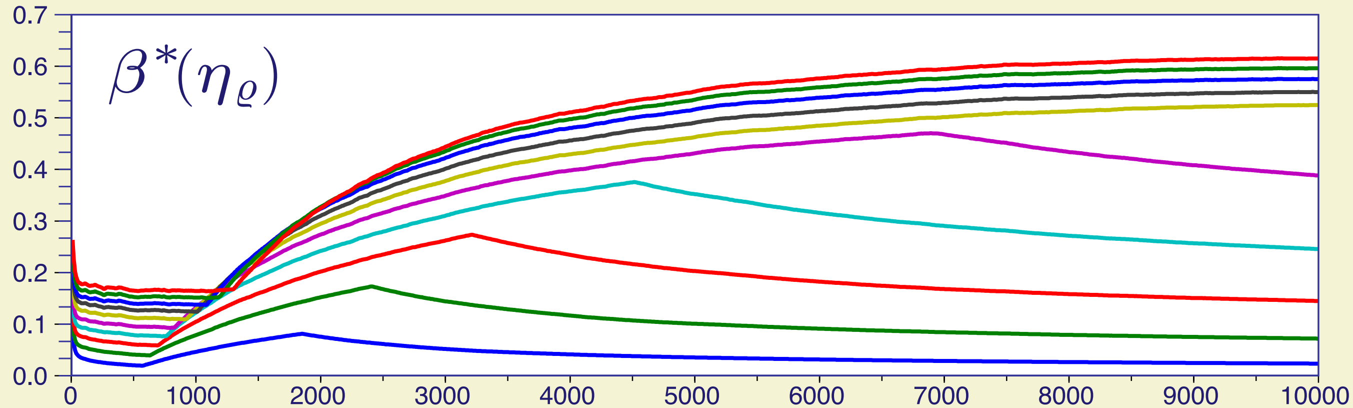
Mismatched Neyman Pearson Hypothesis Testing

X^1 uniform on $[0,1]$, $\quad X^0 = \sqrt[5]{X^1}$

Stochastic Newton-Raphson



False Alarm Error Exponent

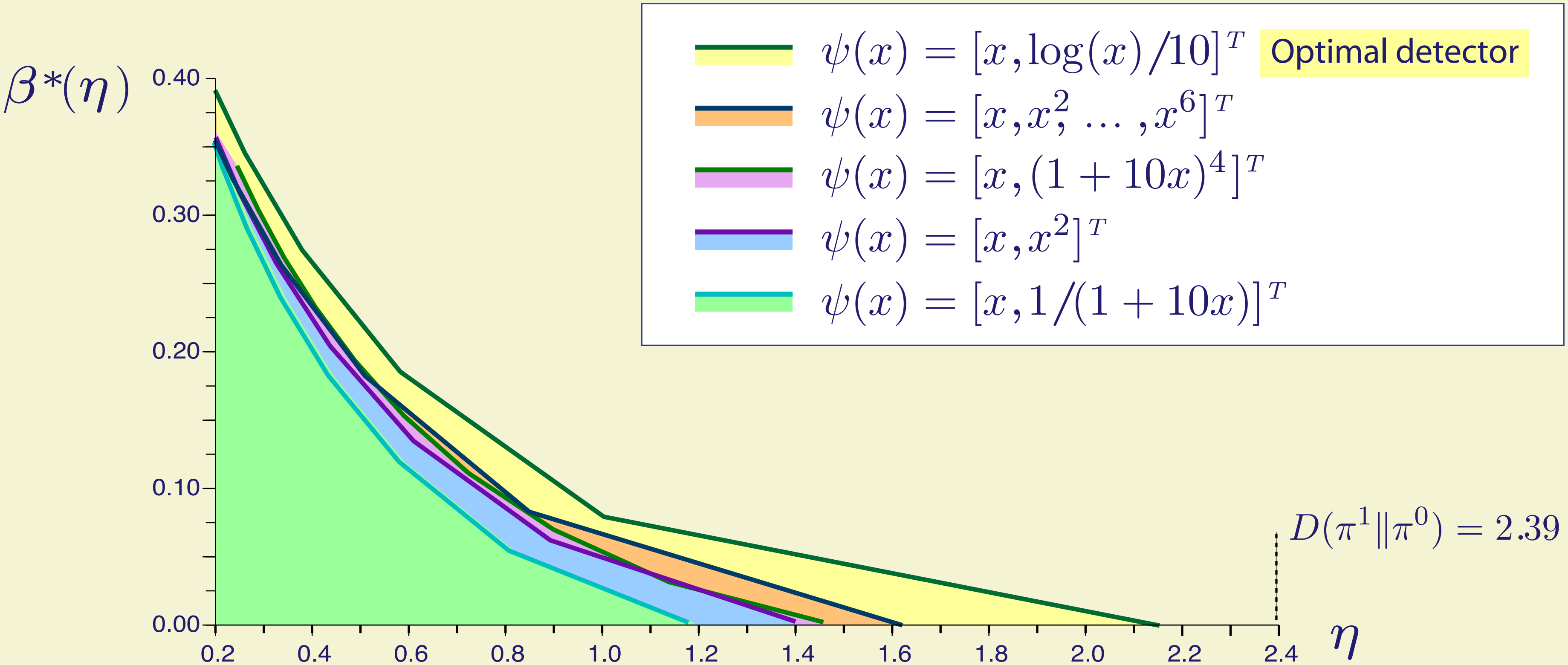


Missed Detection Error Exponent

Parallel simulation for a range of η

Mismatched Neyman Pearson Hypothesis Testing

X^1 uniform on $[0,1]$, $\quad X^0 = \sqrt[5]{X^1}$



Parallel simulation for a range of η

IV
Conclusions

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Geometry based on Sanov’s Theorem combined with Stochastic Approximation provides powerful computational tools

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Future work:

Optimizing the input distribution

Application to MIMO will simultaneously resolve coding and resource allocation. *Extensions to network coding possible*?

Simulation algorithm exhibits high variance.
Application of importance sampling? *Revisit robust approach*?

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