Organized Structures in Strongly Stratified Flows

James J. RILEY[α], Marie-Pascale G. LELONG[β], and Donald N. SLINN[α]

[α] *University of Washington*
Seattle, Washington, USA

[β] *National Center for Atmospheric Research*
Boulder, Colorado, USA

Turbulence and Coherent Structures
O. Métais and M. Lesieur, eds.
Selected papers from "Turbulence 89: organized structures and turbulence in fluid mechanics"
Grenoble, 18-21 September 1989

We report on numerical simulations of internal wave and vortical mode interactions in stably-stratified fluids. Two problems were considered: a wave/vortical mode resonance, and the Taylor-Green problem. In both cases the results of the simulations supported the perturbation theory of Lelong and Riley (1989) for Froude numbers of order one of less. Furthermore, in each case computed in this range of Froude number, the vortical mode exhibited strong instabilities, transferring energy to larger horizontal scales.

1. Introduction

Visual observations from laboratory experiments indicate that stable density stratification can totally alter the dynamics of a turbulent flow. For example, Lin and Pao (1979) observed from visualization studies that, in the later stages of decay of a turbulent wake in a stably-stratified fluid, the wake exhibited large-scale, quasi-horizontal meanders superimposed on the undulations of an internal wave field. (See Figure 1, taken from Hopfinger, 1987). Liu (1980) has observed similar behavior in the later stages of decay of grid turbulence in a stably stratified fluid. More recently, van Heijst and Flór (1989) observed the long-time behavior of a short duration jet in a strongly-stratified fluid and again noted the quasi-horizontal character of the motion.

Riley et al. (1981) have suggested that the presence of quasi-horizontal structures, for which we will use the term vortical modes, is due to the stratification domination of the flow. Introducing a Froude number defined by $F = u'/NL$, where u' is an rms velocity, N is the buoyancy frequency, and L is an integral scale (characterizing the larger-scale structure), they argued that the vortical modes appear when F is small. Furthermore, they offered scaling arguments to explain why the flow should consist of both vortical modes and internal waves when F becomes small, and gave a mathematical decomposition of the velocity field into wave and vortical mode components. Subsequently Lilly (1983) extended these ideas to large-scale geophysical flows (with rotation), and suggested that vortical modes might explain recent atmospheric spectral data (Gage, 1979) at intermediate scales. Müller (1988; see also Müller et al., 1988) has suggested that vortical modes explain the vertical fine-structure observed in the ocean internal wave field. Recently Métais and Herring (1989; Herring and Métais, 1989) performed direct numerical

Fig.1. Wake collapse of a towed sphere in a continuously stratified fluid (from Hopfinger, 1987).

simulations of both decaying homogeneous turbulence and also of forced homogeneous turbulence in strongly-stratified fluids and found the decomposition suggested by Riley et al. to be useful in interpreting the results of the simulations.

In an effort to understand the vortical mode dynamics and their interaction with the internal wave field, Lelong and Riley (1989; see also Lelong, 1989) have recently examined the nonlinear interaction of simple monochromatic internal waves and vortical modes. Taking advantage of the small Froude number, they have formulated the problem using perturbation methods to examine these interactions. In particular, they have concluded the following:

i wave-wave resonant interaction theory extends intact when applied in the presence of weak vortical modes. It is also unaffected by the generalizing assumption that the wave vectors constituting the resonant triad not lie in a vertical plane;

ii a potentially important wave-vortical mode resonant triad exists, in which the vortical mode plays a catalytic role as energy is exchanged between two internal waves of equal frequency;

iii Any two vortical modes can excite a third vortical mode to form a resonant vortical triad. To lowest order, vortical modes satisfy the two-dimensional flow equations in each horizontal plane, while varying in the vertical direction. This verifies rigorously the heuristic suggestions of Riley et al.

The purposes of the work presented in this paper are to (i) examine the validity of the weak interaction theory of Lelong and Riley, and to (ii) investigate further the interactions identified by them. Direct numerical simulations are used to achieve these purposes, with computations being performed for cases predicted by the theory, giving 'exact' solutions to the fully-nonlinear problem. In the next section, we briefly review the theory of Lelong and Riley and discuss their principal results. In the third section, calculations for wave-vortical mode interactions are presented and compared with theory. In addition, simulations are presented for initial conditions consisting only of vortical modes. The behavior of the resulting flows is examined, and the results compared to the scaling analysis suggested by the theory. In the final section, the results are summarized and discussed.

2. Results from Perturbation Analysis

As mentioned in the previous section, vortical modes are visually observed in laboratory experiments when the Froude number based upon the energy containing range, i. e., $F = u'/NL$, is small. This Froude number can be interpreted as the ratio of two time scales: N^{-1}, the buoyancy period, and L/u', an advective time scale. In the experiments, the buoyancy period has thus become small compared to the advection time. F being small is also the usual requirement for linear or weakly nonlinear internal wave theory to be valid.

This suggests the use of multiple time scale perturbation analysis to analyze the dynamics of the internal waves and vortical modes. Lelong and Riley have carried out such an analysis, and begin by writing the velocity field $\vec{u}(\vec{x}, t)$ in the form, as suggested by Lilly (1983):

$$\vec{u} = \nabla \wedge \psi \vec{i}_z + \nabla_H \phi + w \vec{i}_z \,. \qquad (2-1)$$

Here the subscript H denotes the horizontal component. To lowest order the stream function ψ completely determines the vortical mode, while the vertical velocity w (or ϕ, where $w = -\int \nabla^2 \phi \, dz$) specifies the internal wave field. When these expressions are substituted into the Navier-Stokes equations subject to the Boussinesq approximation, an equation for ψ, namely the vertical vorticity equation is obtained while an equation for w results from manipulation of the vertical and horizontal momentum equations.

Velocities are nondimensionalized by u', lengths by L, and time by N^{-1}. (It is assumed that N is uniform in space.) All nondimensionalized dependent variables are expanded in power series of F, and terms of like powers of F are equated. Furthermore, the buoyancy and advective times are now both treated as independent variables. To lowest order (on the short time scale N^{-1}), linear theory is recovered. On this time scale, the vortical modes are steady. The equations at the next order give an $O(F^2)$ correction to these solutions unless a resonance exists, in which case the form of the lowest order solutions has to be reassessed in order for the series expansion to remain valid. The elimination of secular terms from the second order equations yields the evolution of the solutions on the slow, advective time scale. The results depend on the specific problem considered, i. e., on the initial conditions.

2.1. Wave-Wave Interactions

For this case, the initial conditions are taken to be two linear internal waves and a linear vortical mode. The internal waves are arbitrarily oriented, while the vortical mode is assumed to be harmonic in space and is given in terms of a stream function as in Equation (2-1) above. The results of the analysis are an extension of resonant wave interaction theory in the presence of a vortical mode, and the application of resonance theory to triads which are out of the vertical plane. Wave resonant triads are unaffected by the presence of the vortical mode, unless there exists a wave-vortical mode resonance (see the next subsection below). Furthermore, resonant triad analysis is readily extended to wave triads not lying in a vertical plane. The only deviation from the vertical plane case is that the gravity is effectively reduced by the cosine of the angle of the triad-containing plane with respect to the horizontal.

2.2. Wave-Vortical Mode Interactions

In this case, the initial conditions consist of one internal wave (with wave number vector $\vec{\kappa}_1$ and frequency ω) and one vortical mode (with wave number vector $\vec{\kappa}_2$). The analysis predicts that a resonance can occur with a second internal wave (with wave number vector $\vec{\kappa}_3$) of the same frequency if the wave numbers satisfy

$$\vec{\kappa}_1 \pm \vec{\kappa}_2 \pm \vec{\kappa}_3 = 0 . \qquad (2-2)$$

The wave number vectors of the internal waves lie on a vertical cone whose surface is at angle $\theta = \cos^{-1}\omega$. (See Figure 2.) Note that this interaction could act to redistribute energy broadly in wave number space.

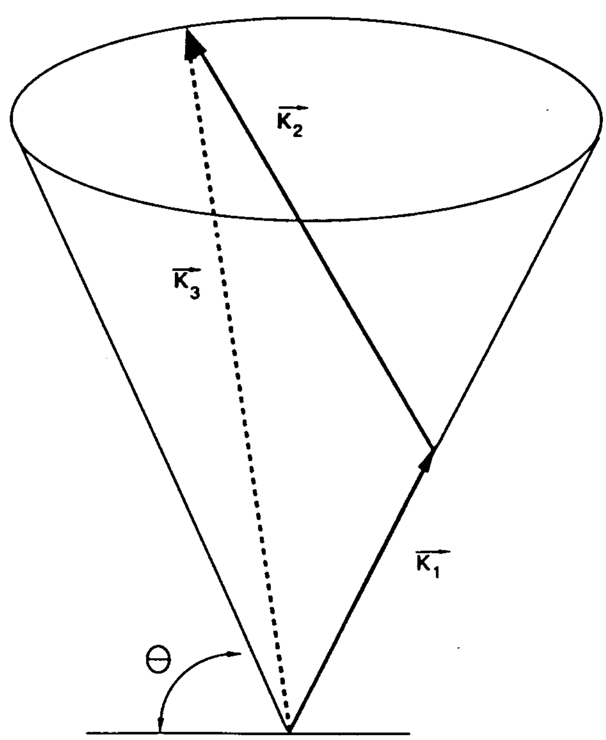

Fig.2. Wave/vortical mode resonant triad.

The nonlinear amplitude equations for this case can be solved analytically, and it is found that the role of the vortical mode is catalytic, i. e., it is needed for the interaction to occur, but it does actively participate in the energy exchange. (This mechanism is reminiscent of the elastic scattering interaction discovered by Phillips, 1968, although the dynamics for the present case are significantly different.) The two wave modes exchange energy harmonically on the slow time scale, the vertical velocity being given by

$$w = \cos\Gamma F t \sin(\vec{\kappa}_1 \cdot \vec{x} - \omega t) - \sin\Gamma F t \cos(\vec{\kappa}_3 \cdot \vec{x} - \omega t)\,. \qquad (2-3)$$

$2\pi/\Gamma F$ is the interaction period, where Γ is given by

$$\Gamma = \frac{B\kappa_1\kappa_3 \sin^2\theta \sin\Delta\gamma}{2}\left\{\cos^2\theta\cos\Delta\gamma + \sin^2\theta\right\}. \qquad (2-4)$$

Here B is the amplitude of the vortical mode and $\Delta\gamma$ is the angle between the horizontal components of the wave number vectors of the two waves.

2.3. Vortical Mode Interactions

In this final case, Lelong and Riley have considered initial conditions consisting of two vortical modes. It is found that all interactions are resonant because the resonance condition imposed on the frequencies is identically satisfied, the frequency of any vortical mode being zero regardless of its wavenumber. As a consequence, the vortical mode equations are fully-nonlinear on the vortical mode time scale (L/u'). It is found that, to lowest order, the vortical mode velocity and pressure satisfy:

$$\frac{\partial}{\partial t}\vec{u}_H + \vec{u}_H \cdot \nabla\vec{u}_H = -\nabla p + R^{-1}\nabla^2\vec{u}_H \qquad (2-5a)$$

$$\nabla \cdot \vec{u}_H = 0\,. \qquad (2-5b)$$

Here $R = u'L/\nu$ is the Reynolds number. Note that these equations describe horizontal motion in each horizontal plane, but with vertical variation retained. This result was suggested by Riley et al. (1981) using heuristic arguments. In addition to the conclusion that the vortical modes satisfy Equation (2-5), it was found that the vortical modes excite internal waves at higher order, with the amplitude of the internal waves scaling as F (or the energy as F^2).

3. Simulations Results

In this section we present results of direct numerical simulations and comparisons with theory for both wave-vortical mode interactions and vortical mode self-interactions. The simulations employ pseudo-spectral numerical methods with leap-frog time-stepping (smoothed every 25 time steps). The wave-vortical mode simulations were performed on $32 \times 32 \times 32$ point computational grids, while the vortical mode interactions used $64 \times 64 \times 64$ point grids. All calculations were performed on an Ardent Titan Mini-Supercomputer Server.

3.1. Wave-Vortical Mode Interactions

As discussed in the previous section, a resonance was found to exist between a vortical mode and two internal waves if the two waves are of the same frequency and the three wave number vectors form a triangle on the surface of a vertically-oriented cone. In order to examine these interactions we have performed direct numerical simulations for a number of different resonance conditions, and for a variety of different Froude numbers. The following case is typical of the results from these simulations, and represents the interaction of an internal wave with a horizontally-shearing current. The latter is a special case of a vortical mode. The initial vertical velocity of the internal wave satisfies

$$w(\vec{x},0) = \cos(\vec{\kappa}_1 \cdot \vec{x}),$$

while the vortical mode is initially given by

$$\vec{u}(\vec{x},0) = \Big(\kappa_{2_2} B \sin(\vec{\kappa}_2 \cdot x), -\kappa_{2_1} B \sin(\vec{\kappa}_2 \cdot x), 0\Big).$$

We choose

$$\vec{\kappa}_1 = (2,0,2),$$

giving a wave propagating at 45^o to the horizontal, and

$$\vec{\kappa}_2 = (-2,2,0).$$

Therefore the resonant wave should be found at

$$\vec{\kappa}_3 = (0,2,2).$$

B is taken to be 2.0, and the Froude number F to be 0.02. From the scaling analysis, this implies that the wave frequency ω in the simulations is $2\pi/F$, while the interaction frequency Γ is 2.0. The perturbation theory indicates that both the kinetic and the potential energy in the initial internal wave ($\vec{\kappa}_1$) should vary with time as $0.5\cos^2(2t)$, and that in the forced wave ($\vec{\kappa}_3$) as $0.5\sin^2(2t)$.

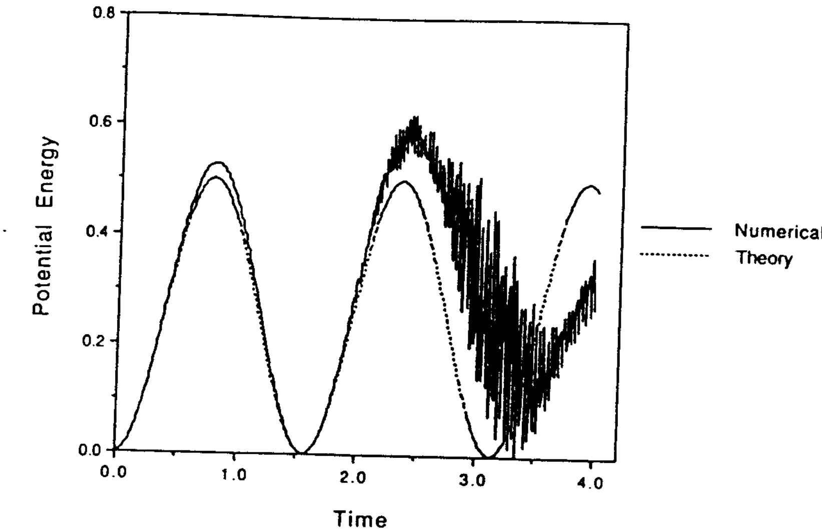

Fig.3. Potential energy in the forced wave $\vec{\kappa}_3$; $F_t = 0.02$.

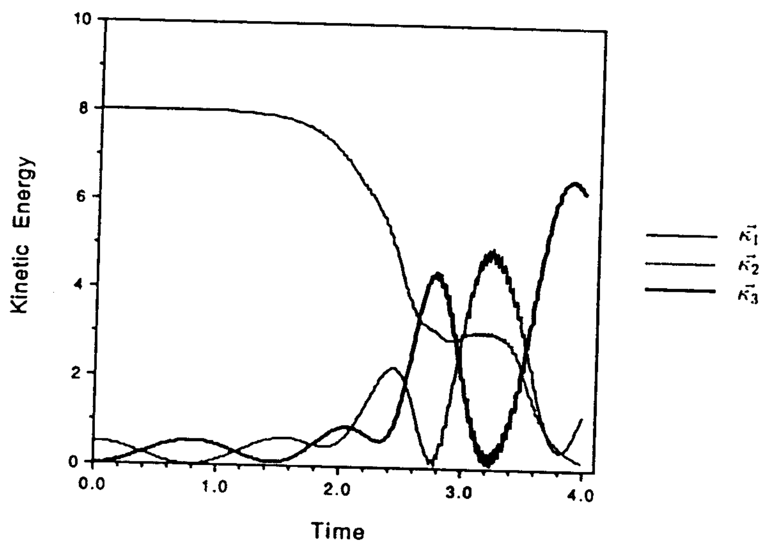

Fig.4. Kinetic energy in all three modes; $F_t = 0.02$.

Figure 3 contains a plot of the potential energy at the forced wave number $\vec{\kappa}_3$ versus time taken from the simulations. Also plotted is the theoretical prediction. We see that the simulations follow the theory fairly closely for a time of about 2 to 2.5, at which point the potential energy in this wave mode begins to oscillate (near the buoyancy frequency), and the results begin to deviate significantly from the theory. More insight into this problem is given by examining the kinetic energy of each mode for this case, as shown in Figure 4. According to the theory, the kinetic energy in the vortical mode should remain uniform in time. Again we see that the simulation results follow the predicted oscillation in the wave kinetic energy up to a time of about 2 to 2.5, at which point the computed solutions rapidly diverge from the theoretical predictions. Furthermore, the kinetic energy in the vortical mode remains approximately constant up to this time, and it then begins to exhibit a large deviation from its initial value.

Visual analysis of the results of this case determined that the horizontally shearing motion was subject to a shear instability which arose dramatically at about a time of 2. Furthermore this instability transferred energy mainly into the vortical mode components at wave numbers $\vec{\kappa}_1$ and $\vec{\kappa}_3$. Prior to this instability, however, the perturbation theory gave accurate predictions of the interactions.

3.2. Vortical Mode Interactions

In order to examine nonlinear vortical mode interactions we have considered initial conditions consisting of a sum of spatially-harmonic vortical modes. The initial velocity field is given by

$$\vec{u}(\vec{x},0) = \cos x_3(\cos x_1 \sin x_2, -\sin x_1 \cos x_2, 0)\,, \qquad (3-1)$$

while the initial perturbation density field is identically zero. For the nonstratified case this initial condition defines the Taylor-Green problem (Taylor and Green, 1937), with the velocity field oriented such that it is initially horizontal. This problem has received much attention in the literature (e. g., Orszag, 1971). The perturbation analysis predicts that as $F \to 0$, the flow field should satisfy Equation (2-5) to lowest order. The exact solution to these equations, satisfying the above initial conditions, is

$$\vec{u}(\vec{x},0) = e^{(-3R^{-1}t)} \cos x_3(\cos x_1 \sin x_2, -\sin x_1 \cos x_2, 0)\,. \qquad (3-2)$$

The nonlinear terms in the momentum equation [Equation (2-5a)] are exactly balanced by the pressure gradient, so that the equations become linear. The streamline pattern remains independent of time as the velocity field decays due to viscosity. This solution is an extension of a well-known solution for the two-dimensional Taylor-Green problem (see, e. g., Staquet, 1985).

We performed a series of simulations for a number of different Froude numbers, using the above initial conditions for each case. For all the cases presented the initial Reynolds number R was fixed at 200. Figure 5 gives a plot of the volume integrated kinetic energy of the horizontal velocity (normalized by its initial value) for these different cases. The case with $F = \infty$ corresponds to the Taylor-Green problem, and agrees with previously published simulation results. We see that increasing the stratification decreases the decay rate, as might be expected. Furthermore as F becomes small, roughly $F \leq 1.0$, the computed solutions

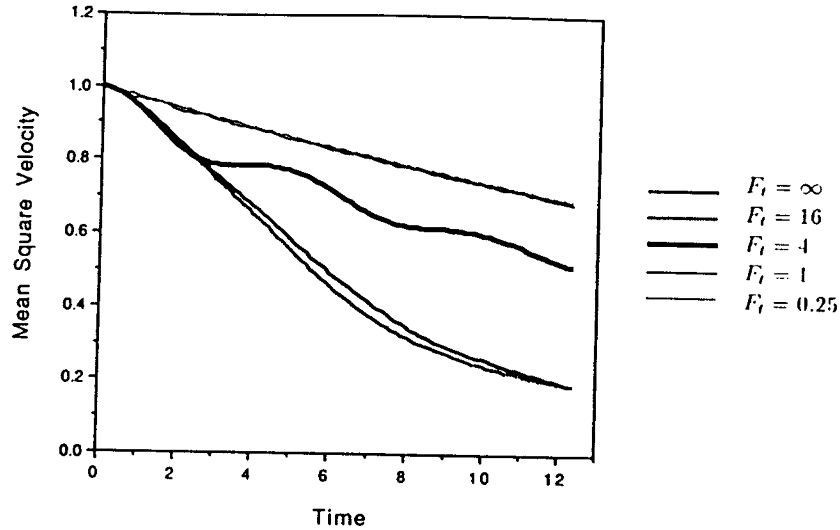

Fig.5. Taylor-Green problem: normalized kinetic energy of horizontal velocity for various Froude numbers.

coincide, as expected from the perturbation theory. Figure 6 gives a plot of the same data on an expanded scale, for the cases with stronger stratification. Also plotted is the result from the theoretical solution, Equation (3-2). We see that the solution to the predicted asymptotic equations agrees very well with the simulation results for $F \leq 1.0$.

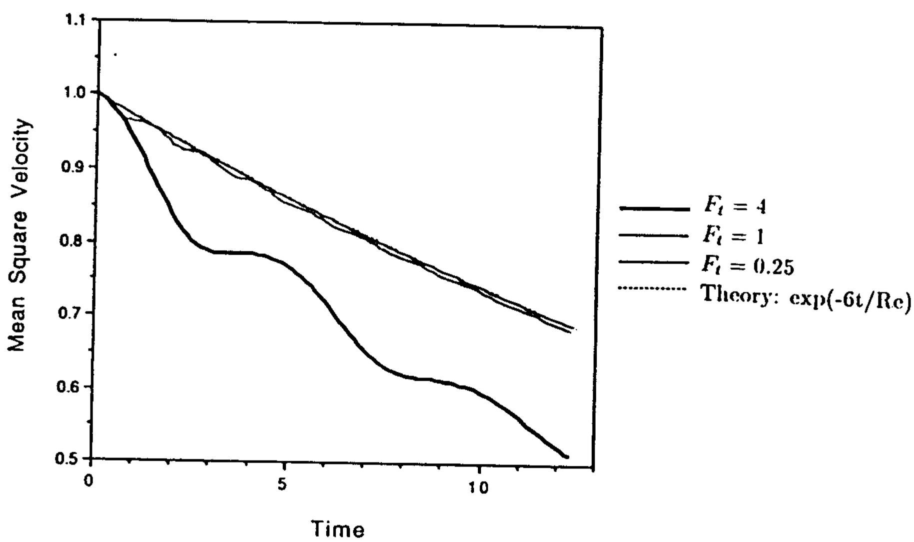

Fig.6. Taylor-Green problem: normalized kinetic energy of horizontal velocity for several Froude numbers – expanded scale.

Figure 7 contains plots of the wave energy for these same cases. The wave energy contains the potential energy plus the wave part of the kinetic energy, as defined by Equation (2-1). Note that the wave energy

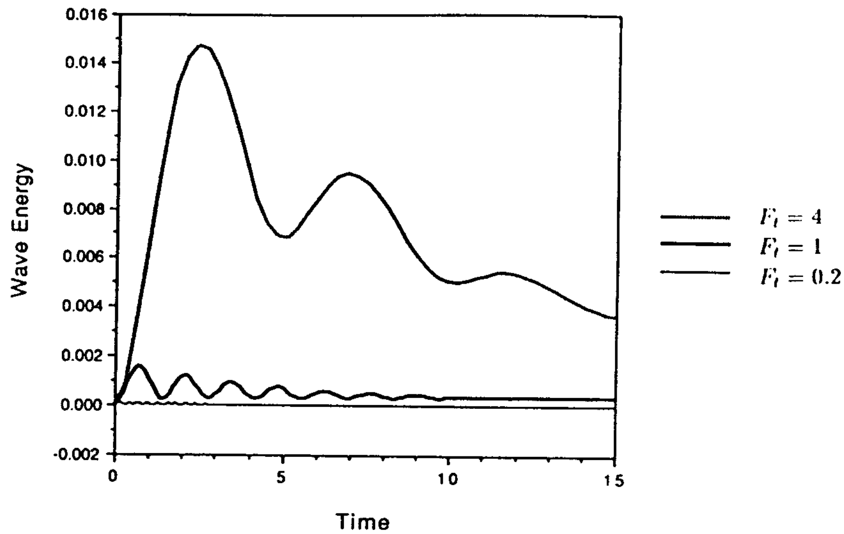

Fig.7. Taylor-Green problem with white noise: wave energy for three different Froude numbers.

decreases significantly as F is decreased. When these results are replotted, scaled with F^{-2}, as suggested by the theory, then the wave energy collapses well for $F \leq 4.0$ (Figure 8). Therefore, for $F \leq 1.0$, the perturbation analysis is consistent with the results for both the horizontal kinetic energy, which consists mainly of the vortical mode, and the wave energy.

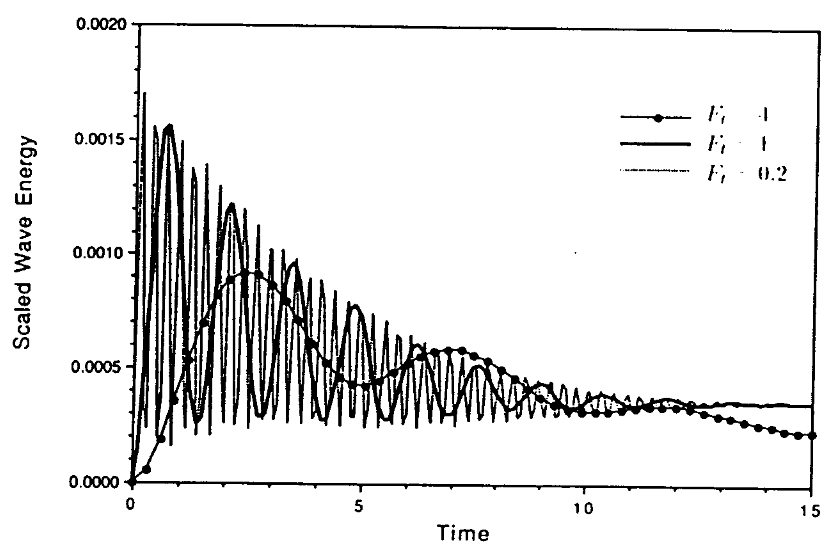

Fig.8. Wave energy scaled by $1/F_t^2$ for three different Froude numbers.

This present case is somewhat degenerate because, as F becomes small, the nonlinear and horizontal pressure gradient terms come into balance, leading to the simple viscous decay given by Equation (3-2). As

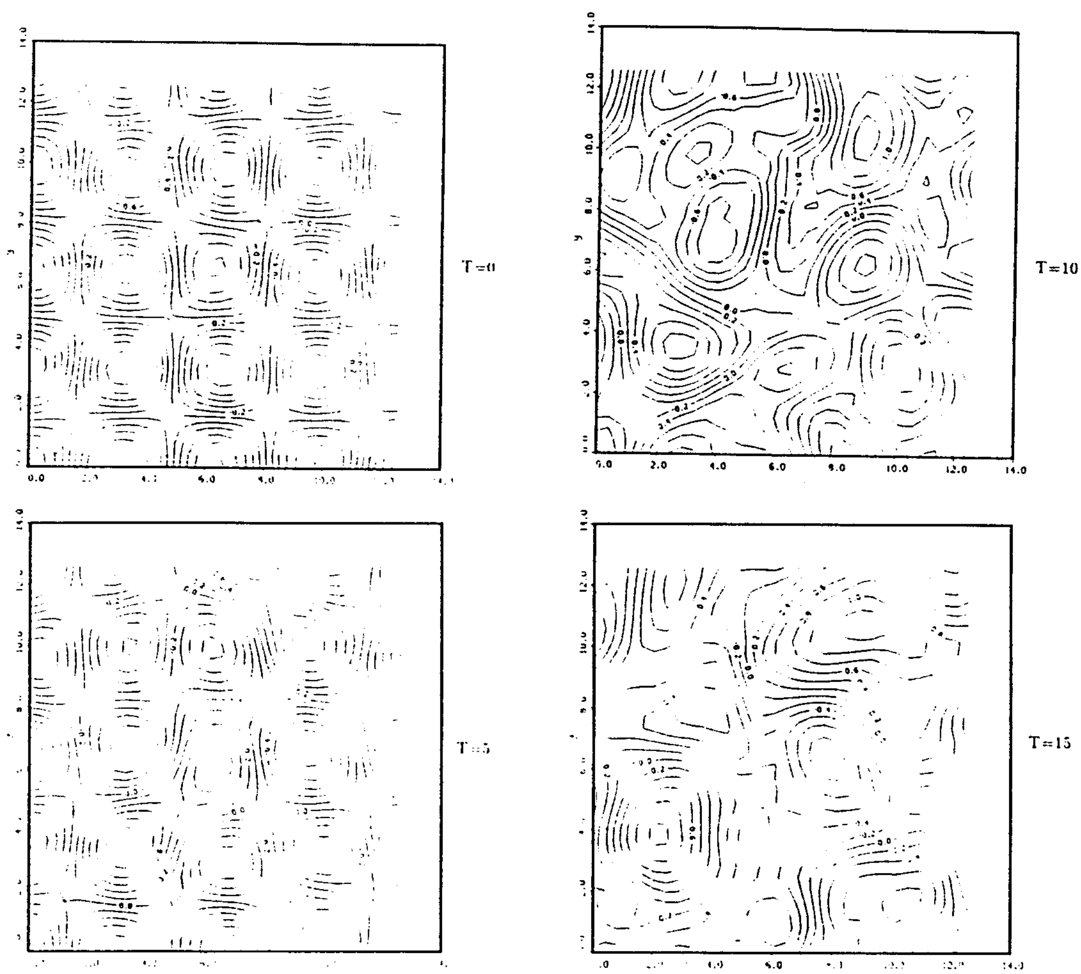

Fig.9. Sequence of constant contours of the stream function in a horizontal plane for $F_t = 0.2$ at $T = 0$, $T = 5$, $T = 10$, and $T = 15$.

mentioned, this is similar to the result for the two-dimensional Taylor-Green problem. It is well-known for the two-dimensional problem, however, that the solution is unstable to subharmonic perturbations. The length scales of the flow continually grow larger as energy is nonlinearly transferred to lower wave numbers. Therefore we also performed a series of simulations for different F using the above initial conditions, Equation (3-1), but with white noise added. Figure 9 gives a sequence of constant contours of ψ in a horizontal plane for the case $F = 1.0$. At this value of stratification these contours approximate streamlines in a horizontal plane. We see that the flow develops nonlinearly in time, as energy is continually transferred to larger scales, reminiscent of the two-dimensional problem. Plate 1 contains plots of ψ in two different horizontal planes at a later time. We see that the two layers have become decoupled, the flow in the two planes being very different. Finally, Figure 10 has plots of the total energy versus time for the different cases computed. Again the results converge for $F \leq 1.0$, consistent with the predictions of the perturbation theory.

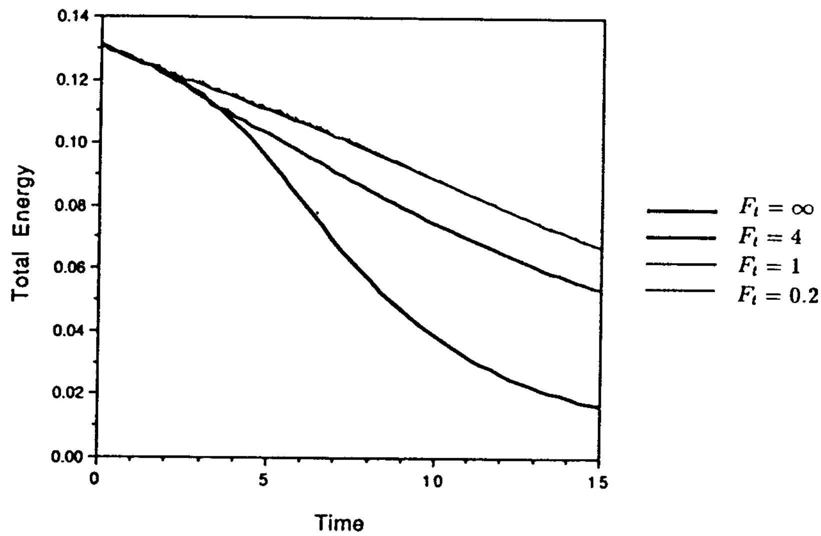

Fig.10. Taylor-Green problem: total energy for several Froude numbers.

4. Conclusions and Discussion

We have reported on direct numerical simulations of internal wave and vortical mode interactions in strongly stratified flows. The objectives are to examine the validity of the weak interaction theory of Lelong and Riley (1989), and to further investigate the interactions identified by them. One of the interactions examined is the resonance of two internal waves and a vortical mode. We have found that the perturbation analysis predicts the interaction very well if the Froude number is small enough, approximately $F \leq 1.0$. From the numerical simulations we have also found, however, that the vortical modes considered were highly unstable, and ultimately experienced breakdown as the fluctuations in the flow increased. In all cases considered for this interaction the vortical mode consisted at least partially of horizontally varying currents with multiple inflection points. These flows satisfy both Rayleigh's and Fjortoft's necessary conditions for instability (Drazin and Reid, 1981). Thus, it is not surprising that they were unstable. The instabilities appear to feed energy into the vortical modes at the same wave number as the internal waves. Once the unstable fluctuations grew to an appreciable amplitude, then the results of the simulations deviated strongly from the perturbation theory predictions.

For the case of vortical mode interactions we considered the Taylor-Green problem. An exact solution to the perturbation equations was found for this case, an extension of a well-known two-dimensional result. The results of the numerical simulations agreed well with this solution for approximately $F \leq 1.0$. Furthermore, the scaling of the potential energy predicted by the theory was also consistent with the simulation results when this condition was satisfied. When white noise was added to the initial Taylor-Green field, for small F the flow exhibited subharmonic instabilities similar to those observed for the two-dimensional Taylor-Green case. Again the perturbation equations as well as the predicted scaling of the potential energy were found to hold for approximately $F \leq 1.0$.

In both cases the simulations emphasize the importance of nonlinearity in the dynamics of the vortical modes. Furthermore, especially in the Taylor-Green case, it is clear that upscale transfer of energy to larger horizontal scales occurs, a phenomena suggested by Lilly (1983), and observed in the laboratory (Lin and Pao, 1979) and in other numerical simulations (Riley et al., 1981; Herring and Métais, 1989). It was also clear that the different horizontal layers tend to become uncorrelated, since the horizontal flow dynamics differ in each layer. How the layers remain weakly coupled, whether through viscous effects or shear instabilities, must surely depend strongly on the Reynolds number.

If vortical modes exist on geophysical scales then their spatial characteristics and dynamical properties become of interest. Clearly, if vortical modes somewhat similar to those considered here were prevalent, then the vortical mode field would be rapidly evolving in time and probably continually be subject to shear instabilities. However if vortical modes existed as two-dimensionally (in the horizontal) stable flows, e. g., in stable rotation satisfying Rayleigh's circulation criterion (Lelong, 1989; McWilliams, 1985), then perhaps they would be more persistent dynamically.

References

Drazin, P. G., and Reid, W. H. . 1981. *Hydrodynamic Stability.* Cambridge University Press.

Gage, K. S. 1979. "Evidence for a $k^{-5/3}$ Law Inertial Range in Mesoscale Two-Dimensional Turbulence", *J. Atmos. Sci.*, **36**, 1950-1954.

Herring, J. R., and Métais, O. 1989. "Numerical Experiments in Forced Stably-Stratified Turbulence", *J. Fluid Mech.*, **202**, 97-115.

Hopfinger, E. J. 1987. "Turbulence in Stratified Fluids: A Review", *J. Geophys. Res.*, **92**(C5), 5287-5304.

Lelong, M.-P. G. 1989. *Weakly-Nonlinear Internal Wave/Vortical Mode Interactions in Stably-Stratified Flows*, Ph. D. Thesis, University of Washington.

Lelong, M.-P. G., and Riley, J. J. 1989. "Internal Wave/Vortical Mode Interactions in a Stably-Stratified Flow", in preparation.

Lilly, D. K. 1983. "Stratified Turbulence and the Mesoscale Variability of the Atmosphere", *J. Atmos. Sci.*, **40**, 749-761.

Lin, J.-T., and Pao, Y.-H. 1979. "Wakes in Stratified Fluids", *Ann. Rev. Fluid Mech.*, **11**, 317-338.

Liu, H.-T. 1980. Private communication.

McWilliams, J. C. 1985. "Submesoscale Coherent Vortices in the Ocean", *Rev. Geophys.*, **23**, 165-182.

Métais, O., and Herring, J. R. 1989. "Numerical Simulations of Freely Evolving Turbulence in Stably-Stratified Fluids", *J. Fluid Mech.*, **202**, 117-148.

Müller, P. 1988. "Vortical Motions", in *Small-Scale Turbulence and Mixing in the Ocean*, ed. by J. C. J. Nihoul and B. M. Jamart, Elsevier, 285-302.

Müller, P., Lien, R.-C., and Williams, R. 1988. "Estimates of Potential Vorticity at Small Scales in the Ocean", *J. Phys. Ocean.*, **18**(3), 401-416.

Orszag, S. A. 1971. "Numerical Simulation of Incompressible Flows Within Simple Boundaries: Accuracy", *J. Fluid Mech.*, **49**(1), 75-112.

Phillips, O. M. 1968. "The Interaction Trapping of Internal Gravity Waves", *J. Fluid Mech.*, **34**, 407-416.

Riley, J. J., Metcalfe, R. W., and Weissman, M. A. 1981. "Direct Numerical Simulations of Homogeneous Turbulence in Stably-Stratified Fluids", in *Proc. AIP Conf. on Nonlinear Properties of Internal Waves*, ed. by B. J. West, 79-112.

Staquet, C. 1985. *Étude Numérique de la Turbulence Bidimensionnelle Homogène et Cisaillée*. Thèse Docteur de l'Institut National Polytechnique de Grenoble.

Taylor, G. I., and Green, A. E. 1937. "Mechanisms of the Production of Small Eddies From Large Ones", *Proc. Roy. Soc.*, A **158**, 499-521.

van Heijst, G. J. F., and Flór, J. B. 1989. "Dipole Formation and Collisions in a Stratified Fluid", *Nature*, **340**(6230), 212-215.

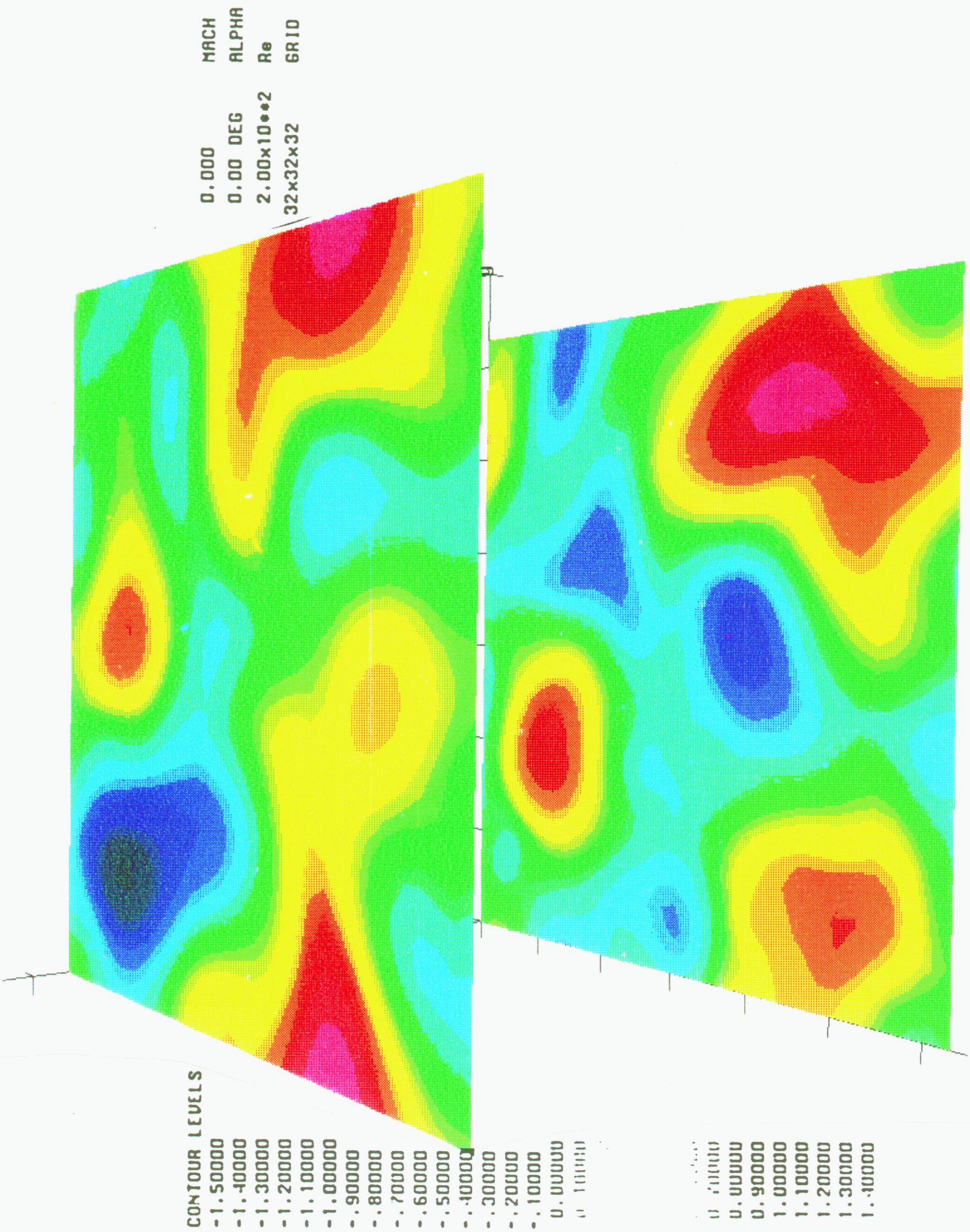

Plate 1. Constant contours of the stream function in two different horizontal planes for $F_t = 0.2$ at $T = 15$.