Chapter 12 Exercises. Linear Algebra.

1. Let A and B be the following matrices:

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -2 \\ 4 & -6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 2 \\ 6 & -4 \end{bmatrix}$$

   Compute the product $C = AB$.

2. Determine mathematically whether the set of vectors:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix} \right\}$$

   forms a basis for $\mathbb{R}^3$.

3. Change of Bases.

   (a) Suppose the representation of $\mathbf{x}$ in terms of the standard basis is $\mathbf{x} = [2, 1]^t$. Calculate the representation of $\mathbf{x}$ in terms of the basis: $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = [2, 4]^t$ and $\mathbf{v}_2 = [2, -4]^t$.

   (b) Calculate the change of basis matrix (call it S) that changes the coordinate system from one using the standard basis to one using the basis $\mathcal{B}$?

   (c) Explain how the coordinate system change defined by S involves scaling and rotating.

   (d) What happens to the unit square, that is, the square with corners at $[0, 0], [1, 0], [0, 1], [1, 1]$ after changing the coordinate system using S?

4. Use the change of basis matrix S from the previous problem and a diagonal matrix to change the representation of the vector $\mathbf{x} = [2, 1]^t$ from the standard basis to the basis $\mathcal{B}$ in the previous problem and do a point-wise multiplication of the new representation with the vector $\mathbf{y}$ whose representation in terms of $\mathcal{B}$ is $[3, 2]^t$. Following that, change the representation of the pointwise multiplication back to the standard basis. Write down a general formula for doing this calculation.

5. Use the law of cosines to show that if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then $|\mathbf{x}^t\mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\|\cos(\theta)$.

6. Construct a unit norm vector from the vector $[1, 1]^t$.

7. Find a relationship between the distance between two unit norm vectors and the cosine of the angle between them. Demonstrate the relationship using the vectors $\mathbf{x}_1 = [1, 0]^t$ and $\mathbf{x}_2 = \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]^t$.