Math Review Solutions.
Linear Algebra Problem Set 2.

1. The eigenvalues and eigenvectors satisfy the equation $\mathbf{A}\mathbf{v}_k = \lambda_k \mathbf{v}_k$ for $k = 1, 2$. This is equivalent to $(\mathbf{A} - \lambda_k \mathbf{I})\,\mathbf{v}_k = \mathbf{0}$. Dropping the explicit reference to k and writing this out element-wise yields

$$\begin{bmatrix} \frac{5}{2} - \lambda & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} - \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Multiplying the top row of $\mathbf{A} - \lambda\mathbf{I}$ by $\frac{3}{2}$ and the bottom row by $\frac{5}{2} - \lambda$ and doing some algebra leads to the equation

$$\left[\left(\lambda - \frac{5}{2}\right)^2 - \frac{9}{4}\right] v_2 = 0.$$

Assume that $v_2 \neq 0$. Then, it must be true that $\left(\lambda - \frac{5}{2}\right)^2 - \frac{9}{4} = 0$. More algebra yields solutions $\lambda = 4$ and $\lambda = 1$. These are the eigenvalues. To find the corresponding eigenvectors, note that:

$$\lambda = 1 \implies \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

There are infinitely many solution satisfying $v_1 = v_2$. For example, $\mathbf{v}_1 = [1, 1]^t$ is an eigenvector. One can check by verifying that $\mathbf{A}\mathbf{v}_1 = \mathbf{v}_1$. Similarly, letting $\lambda = 4$ and solving implies that $v_1 = -v_2$ so, for example, $\mathbf{v}_2 = [1, -1]^t$ is an eigenvector.

2. To diagonalize, the eigenvectors must be unit norm so redefine $\mathbf{v}_1$ and $\mathbf{v}_2$ by taking $\mathbf{v}_1 = \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]^t$ and $\mathbf{v}_2 = \left[\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right]^t$. Then, taking $\mathbf{V} = [\mathbf{v}_1 | \mathbf{v}_2]$ and

$$\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

one can verify that $\mathbf{V}^t\mathbf{A}\mathbf{V} = \mathbf{D}$.

3. The change of basis matrix is $\mathbf{V}^t$. Recall that $cos\left(\frac{\pi}{4}\right) = sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. Therefore,

$$\mathbf{V}^t = \begin{bmatrix} cos\left(\frac{\pi}{4}\right) & sin\left(\frac{\pi}{4}\right) \\ -sin\left(\frac{\pi}{4}\right) & cos\left(\frac{\pi}{4}\right) \end{bmatrix} \tag{C.1}$$

is a rotation matrix so the change of basis does not change the $\mathcal{L}_2$ unit circle. $\mathbf{V}$ will rotate the $\mathcal{L}_1$ "unit circle" by an angle of $\frac{\pi}{4}$ as shown in Fig. C.3.

4. To simultaneously diagonalize $\mathbf{A}$ and $\mathbf{B}$, note that $\mathbf{A}$ plays the role of $\mathbf{C}$ and $\mathbf{C}_n$ in the text. The procedure is as follows:

- Diagonalize $\mathbf{A}$. Divide by the eigenvalues to map $\mathbf{A}$ to the identity matrix: $\mathbf{D}^{-\frac{1}{2}}\mathbf{V}^t\mathbf{A}\mathbf{V}\mathbf{D}^{-\frac{1}{2}} = \mathbf{I}$,

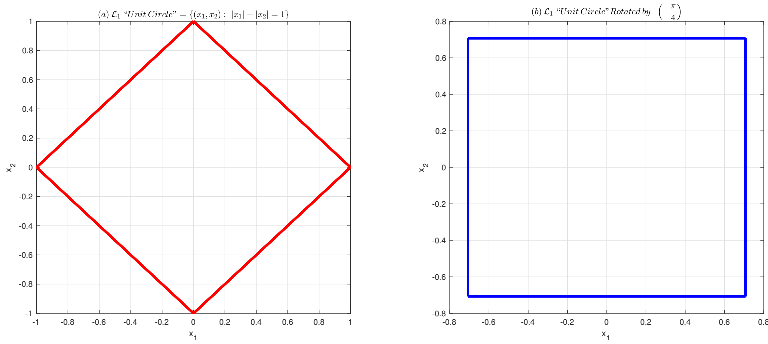

Figure C.3 The $\mathcal{L}_1$ unit circle, $\mathcal{C} = \{x_1, x_2 \in \mathbb{R} : |x_1| + |x_2| = 1\}$ in (a). The effect of multiplying all the points on $\mathcal{C}$ by the matrix $\mathbf{V}$ in Eq. $C.2$.

- Apply the same transformation to $\mathbf{B}$: $\mathbf{B}_n = \mathbf{D}^{-\frac{1}{2}}\mathbf{V}^t\mathbf{B}\mathbf{V}\mathbf{D}^{-\frac{1}{2}}$,
- Diagonalize $\mathbf{B}_n$ to so $\mathbf{U}^t\mathbf{B}_n\mathbf{U} = \mathbf{D}_n$ where $\mathbf{D}_n$ is a diagonal matrix.
- Take $\mathbf{W}^t = \mathbf{U}^t\mathbf{D}^{-\frac{1}{2}}\mathbf{V}^t$.

Then $\mathbf{W}^t\mathbf{A}\mathbf{W} = \mathbf{I}$ and $\mathbf{W}^t\mathbf{B}\mathbf{W} = \mathbf{D}_n$ so $\mathbf{W}$ simultaneously diagonalizes $\mathbf{A}$ and $\mathbf{B}$.

The specific numbers are as follows: $\mathbf{V}$ and $\mathbf{D}$ are given in exercise 2,

$$\mathbf{D}^{-\frac{1}{2}} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \mathbf{B}_n = \begin{bmatrix} 5.866 & -0.250 \\ -0.250 & 1.034 \end{bmatrix}, \mathbf{U} = \begin{bmatrix} -0.052 & -0.999 \\ -0.999 & 0.052 \end{bmatrix} \text{ and } \mathbf{D}_n = \begin{bmatrix} 1.021 & 0.000 \\ 0.000 & 5.889 \end{bmatrix}.$$

Finally,

$$\mathbf{W} = \begin{bmatrix} -0.390 & -0.688 \\ 0.317 & -0.724 \end{bmatrix}.$$