Math Review Solutions.
Linear Algebra Problem Set 1.

1. $C = AB = \begin{bmatrix} 7 & -10 \\ 3 & 14 \\ -16 & 32 \end{bmatrix}$.

2. Since $[2, 3, -2]^T = -[-2, -3, 2]^T$, they are linearly dependent. Hence, they are not a basis for $\mathbb{R}^3$.

3. (a) Solve $[\mathbf{v}_1|\mathbf{v}_2]\,\mathbf{a} = \mathbf{x}$ for $\mathbf{a}$, we get $\mathbf{a} = \left[\frac{5}{8}, \frac{3}{8}\right]^T$. Hence $\mathbf{x} = \frac{5}{8}\mathbf{v}_1 + \frac{3}{8}\mathbf{v}_2$. Therefore,

$$\mathbf{x} = \left[\frac{5}{8}, \frac{3}{8}\right]^T \qquad \text{if the basis } \{\mathbf{v}_1, \mathbf{v}_2\} \text{ is used.}$$

$$\mathbf{x} = [2, 1]^T \qquad \text{if the standard basis is used.}$$

Sometimes this is written as $\mathbf{x} = [2, 1]^T$ and $\mathbf{x}' = \left[\frac{5}{8}, \frac{3}{8}\right]^T$ The new representation is depicted in Fig. $C.1$

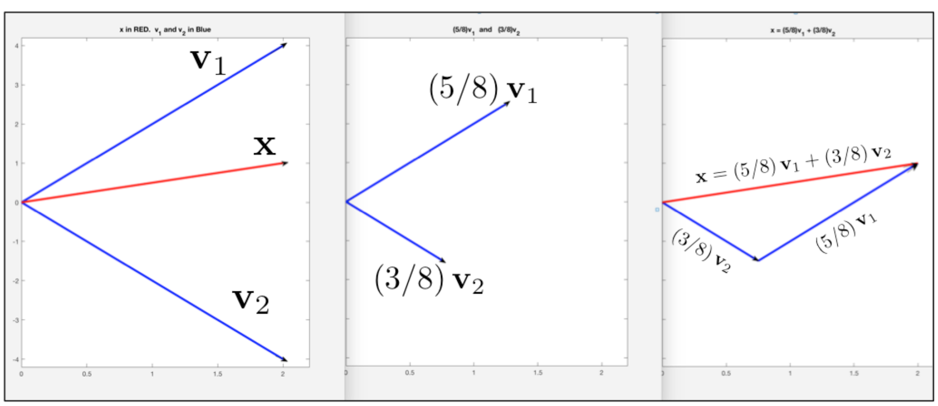

Figure C.1 Representing a vector $\mathbf{x}$ with representation $[2, 1]^T$ in terms of the standard basis, shown in red in terms of the basis $\mathbf{v}_1$ and $\mathbf{v}_2$, shown in blue.

(b) Suppose $\mathbf{x} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2$ be any vector in $\mathbb{R}^2$. x_1 and x_2 are the coefficients of $\mathbf{x}$ with respect to the standard basis. Then, as in the previous problem, one wants to write $\mathbf{x}'$ $=a_1\mathbf{v}_1 + a_2\mathbf{v}_2$. Again as in the previous problem, this requires solving the linear system $[\mathbf{v}_1|\mathbf{v}_2]\,\mathbf{a} = \mathbf{x}$ so, since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, $\mathbf{a} = [\mathbf{v}_1|\mathbf{v}_2]^{-1}\,\mathbf{x}$. Therefore, the change of basis matrix is $\mathbf{S} = [\mathbf{v}_1|\mathbf{v}_2]^{-1}$ since $\mathbf{S}$ maps the representation in terms of the standard basis to the representation in terms of $\mathcal{B}$. Calculating the inverse yields

$$S = [\mathbf{v}_1|\mathbf{v}_2]^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix}$$

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(c) The coefficients of any $\mathbf{x}$ written in terms of $\mathcal{B}$ will be smaller than the coefficients written in terms of the standard basis because both $\mathbf{v}_1$ and $\mathbf{v}_2$ have larger norms than $\mathbf{i}_1$ and $\mathbf{i}_2$. $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 2\sqrt{5}$. The rotation of the coordinate system is not uniform because the angle between $\mathbf{i}_1$ and $\mathbf{i}_2$ is $\frac{\pi}{2}$ but the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ is the inverse cosine of $\frac{\mathbf{v}_1^T\mathbf{v}_2}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|} = cos\,(\theta)$ which is about $2.2143 = \frac{\pi}{1.4188}$ radians, which is approximately 81^o.

(d) The corners of the unit square are transformed as follows:

$$\mathbf{S}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \mathbf{S}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

$$\mathbf{S}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} \\ -\frac{1}{8} \end{bmatrix} \qquad \mathbf{S}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\ \frac{1}{8} \end{bmatrix}$$

which is a rectangle in the new coordinate system since the matrix

$$\mathbf{C} = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix}$$

is orthogonal. Therefore, the area of the transformed unit square in the new coordinate system is $HW = \sqrt{4\left(\frac{1}{4}\right)^2\left(\frac{1}{8}\right)^2} \approx 0.0625$ as shown in Fig. C.2.

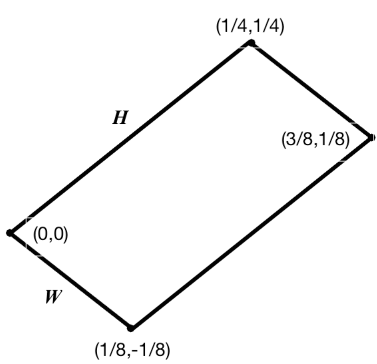

Figure C.2 The Unit Square in standard coordinates is transformed to the unit shown here. The original unit square has area 1 whereas the transformed unit square has area $\frac{1}{8}$.

(4) The vector $\mathbf{x}' = \mathbf{S}^{-1}\mathbf{x}$ is the representation of $\mathbf{x}$ with respect to the basis $\mathcal{B}$. Define

$$\mathbf{D} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then $\mathbf{p} = \mathbf{D}\mathbf{x}'$ represents the pointwise multiplication. Finally, $\mathbf{Sp} = \mathbf{SDS}^{-1}\mathbf{x}$ represents the entire process.

5. The law of cosines says that given a triangle with a, b, c as the lengths of the three sides, θ as the angle between the sides a, b, we have $c^2 = a^2 + b^2 - 2ab\cos\theta$. Use $\mathbf{x}, \mathbf{y}$ as two sides of the triangle, we have $\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta$. Since $\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 - 2\mathbf{x}^T\mathbf{y} + \|\mathbf{y}\|^2$, we have $\mathbf{x}^T\mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|\cos\theta$.

6. $\frac{1}{\sqrt{2}}[1, 1]^T = \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$.

7. Let $\mathbf{x}, \mathbf{y}$ be two unit norm vectors. Since $\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta = 2 - 2\cos\theta$, we have $\cos\theta = 1 - \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|^2$. If the two vectors are $[1, 0]^T$, $\left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]^T$, we have $\cos\theta = \frac{1}{2}$, $1 - \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|^2 = \frac{1}{2}$.