1. The properties of a norm are:

   (a) $\mathcal{N}(\mathbf{x}) \geq 0$ and $\mathcal{N}(\mathbf{x}) = 0$ if and only if $\mathbf{x} = 0$.

   (b) $\mathcal{N}(a\mathbf{x}) = |a|\mathcal{N}(\mathbf{x})$.

   (c) $\mathcal{N}(\mathbf{x}+\mathbf{y}) \leq \mathcal{N}(\mathbf{x}) + \mathcal{N}(\mathbf{y})$

   Since $\mathbf{A}$ is positive definite, it has full rank so $\mathbf{x}^t\mathbf{A}\mathbf{x} \geq 0$ and $\mathbf{x}^t\mathbf{A}\mathbf{x} = 0$ if and only if $\mathbf{x} = 0$ so the first property holds.

   If $a \in \mathbb{R}$, then $(a\mathbf{x})^t \mathbf{A}(a\mathbf{x}) = a^2(\mathbf{x}^t\mathbf{A}\mathbf{x}) \neq |a|(\mathbf{x}^t\mathbf{A}\mathbf{x})$ unless $a = 0$. Therefore, the second property does not hold so it is not a norm. NOTE: This is the only part of the answer that is required.

2. $\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\| + \|\mathbf{y}\| - 2\mathbf{x}^t\mathbf{y} = 2 - 2cos(\theta)$.

   Since $\theta = \frac{\pi}{3}$, the relationship implies that
   $\|\mathbf{x}-\mathbf{y}\|^2 = 2 - 2cos\left(\frac{\pi}{3}\right) = 1$.

   On the other hand, direct calculation shows that
   $\|\mathbf{x}-\mathbf{y}\|^2 = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$.

3. (a) Diagonalizing a matrix $\mathbf{A}$ means finding an invertible matrix $\mathbf{S}$ and a diagonal matrix $\boldsymbol{\Lambda}$ such that $\mathbf{A} = \mathbf{S}\boldsymbol{\Lambda}\mathbf{S}^{-1}$.

   (b) $\mathbf{C}$ is symmetric because $\mathbf{C}^t = \left(\mathbf{U}\Lambda\mathbf{U}^t\right)^t = \left(\mathbf{U}^t\right)^t \Lambda^t\mathbf{U}^t = \mathbf{U}\Lambda\mathbf{U}^t = \mathbf{C}$.

   (c) $\Lambda^{-\frac{1}{2}}\mathbf{U}^t\mathbf{C}\mathbf{U}\Lambda^{-\frac{1}{2}} = \Lambda^{-\frac{1}{2}}\Lambda\Lambda^{-\frac{1}{2}} = \mathbf{I}$.

4. The problem requires calculating c_1 and c_2 such that $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{S}\mathbf{c}$ where the columns of $\mathbf{S}$ are $\mathbf{v}_1$ and $\mathbf{v}_2$. This requires solving a linear system using a sequence of calculations such as elimination of variables by taking linear combinations of rows of $\mathbf{S}$. It is conceptually equivalent to calculating $\mathbf{S}^{-1}$ and then $\mathbf{S}^{-1}\mathbf{x}$ to get $\mathbf{c}$. The solution is $\mathbf{S}^{-1}\mathbf{c} = (5/8, 3/8)^t$.

5. This problem is much easier because $\mathbf{S}^{-1} = \mathbf{S}^t$ since $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthonormal. Therefore, no arithmetic operations are required to compute $\mathbf{S}^{-1}$. The same fact can be seen as a consequence of the orthonormality of $\mathbf{v}_1$ and $\mathbf{v}_2$ by noting that $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \implies c_k = \mathbf{v}_k^t\mathbf{x}$ for $k = 1, 2$.