Math Review Homework.
Probability Problem Set 2.

1. Show that Bin$(m|N,p)$ is a PDF, that is, that $\sum_{m=0}^{N} \text{Bin}\,(m|N,p) = 1$.

2. Show that univariate and multivariate Gaussians are conjugate priors on the mean of univariate and multivariate Gaussians.

3. Calculate $f\,(\mathbf{p}|\mathbf{a})$ that is proportional to the posterior if the prior is Dir$(\mathbf{p}|\,(2,4,3))$ and the likelihood is Mult$((8,3,2)\,|\mathbf{p})$

4. Derive the log-likelihood function of a Gaussian given a set of independent samples $\mathcal{X} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$

5. Derive the log-likelihood function of a Multinomial given a set of independent samples $\mathcal{X} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ where each $\mathbf{x}_n$ is of the form

$$x_{n,m} = 1 \text{ if the outcome of experiment } n \text{ is the } m^{th} \text{ outcome} \tag{B.1}$$

$$= 0 \text{ otherwise.} \tag{B.2}$$

6. Suppose

$$\begin{aligned}
&\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_M\} \subset \mathbb{R}^B \text{ is a set of parameter vectors,}\\
&\mathcal{X} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\} \subset \mathbb{R}^B \text{ is a set of data vectors,}\\
&\mathcal{Y} = \{y_1, y_2, \ldots, y_N\} \subset \mathbb{R},\\
&\mathbf{w} \in \mathbb{R}^M.
\end{aligned}$$

Define

$$\begin{aligned}
K\,(\mathbf{x}, \mathbf{c}) &= e^{-\frac{(\mathbf{x}-\mathbf{c})^t(\mathbf{x}-\mathbf{c})}{2\sigma^2}} \text{ and}\\
f\,(\mathbf{x}_n; \mathcal{C}, \mathbf{w}, \sigma) &= \sum_{m=1}^{M} w_m K\,(\mathbf{x}_n, \mathbf{c}_m)\\
J &= \sum_{n=1}^{N} \left(y_n - f\,(\mathbf{x}_n; \mathcal{C}, \mathbf{w}, \sigma)\right)^2.
\end{aligned}$$

Use gradient descent to define update formulas for $\mathbf{c}_m$ and σ to minimize J.

7. Solve the problem: Minimize $J\,(x,y) = 4x^2 + 9y^2$ subject to $x^2 + y^2 = 1$.